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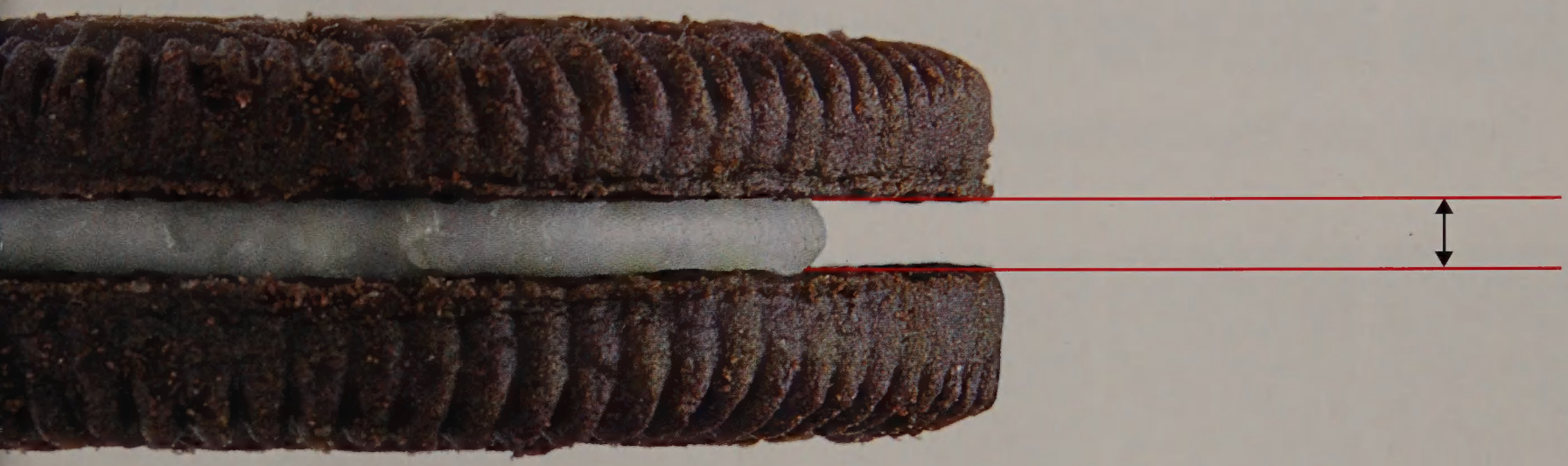

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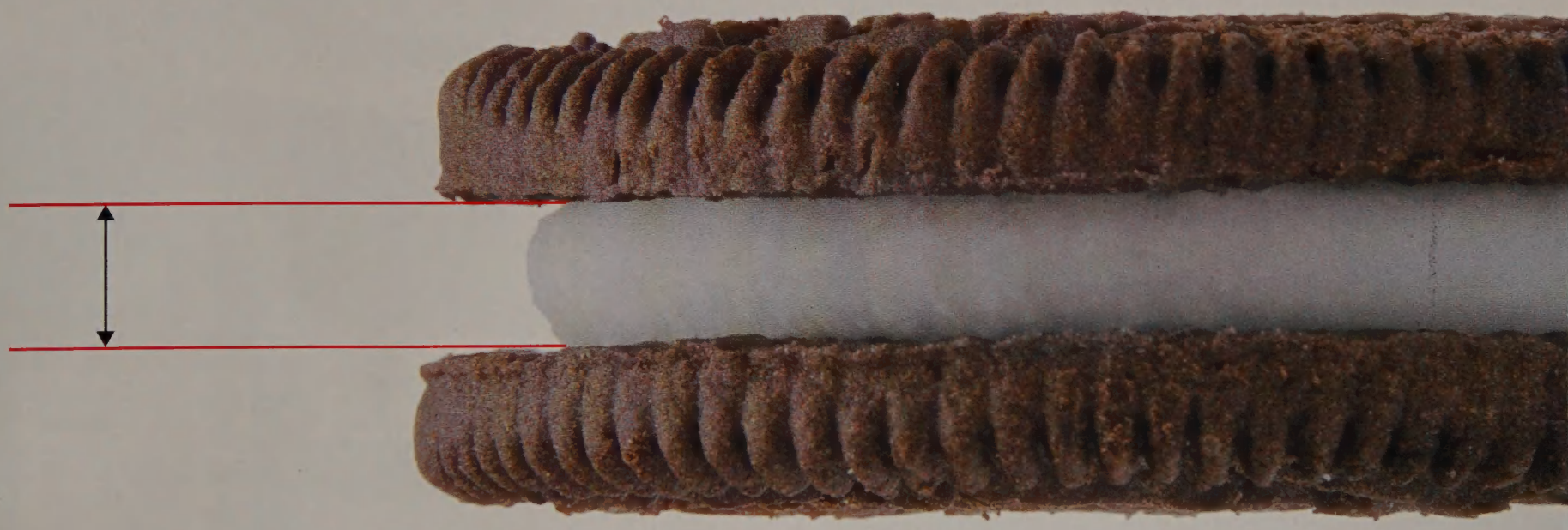

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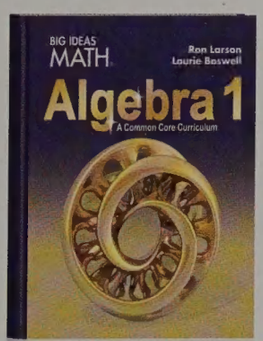

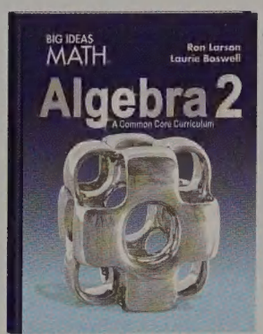

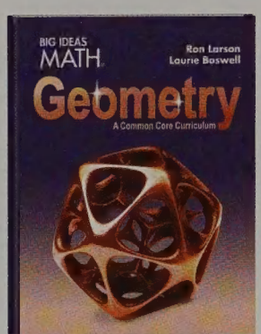

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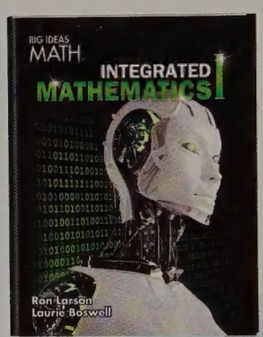

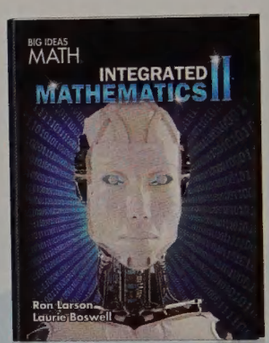

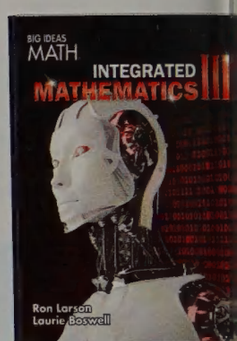

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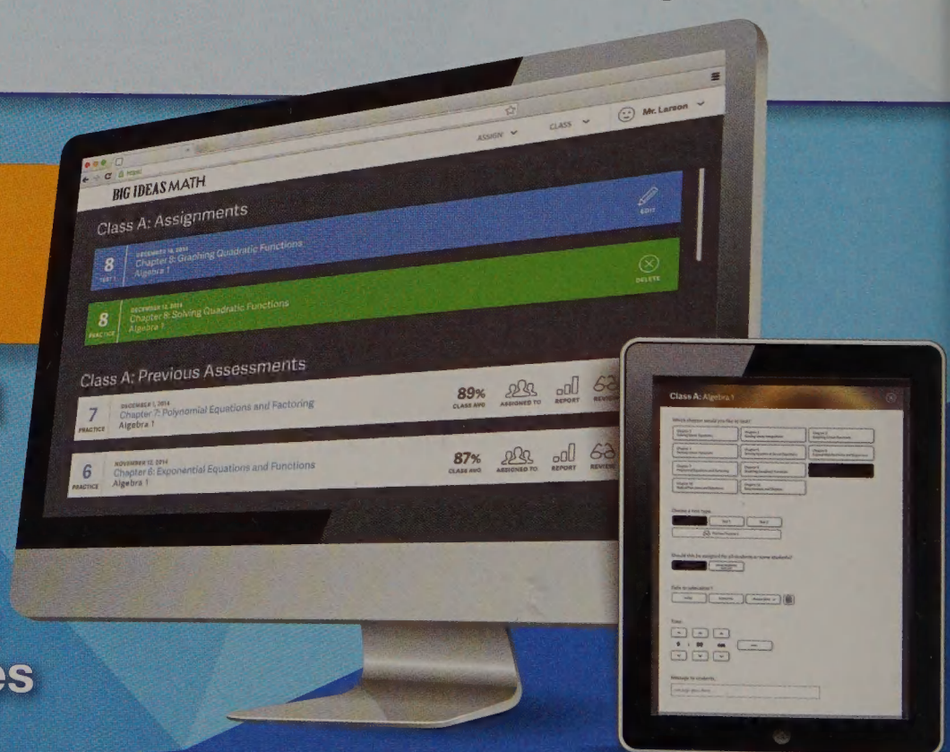
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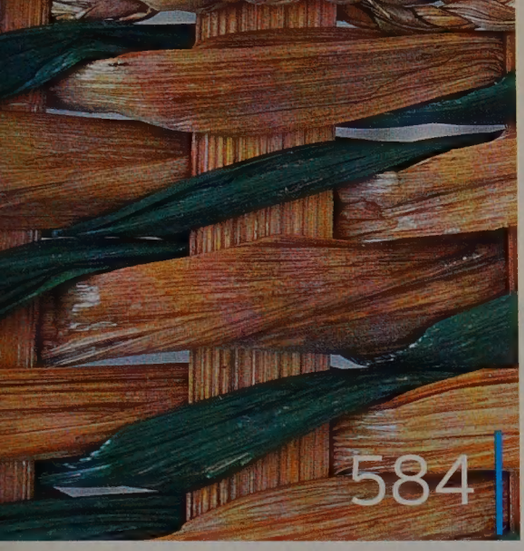

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Should we believe media reports? A well-publicized conclusion led authors Jeremy F. Strayer and Michael Todd Edwards to begin their own investigation of the Cookie Conundrum. Their students replicate and critique the original study and then formulate questions and interpret results. Statistical analysis confirms it—we're sampling "Smarter Cookies" (pp. 608–15).

COVER: CHUNG LEE

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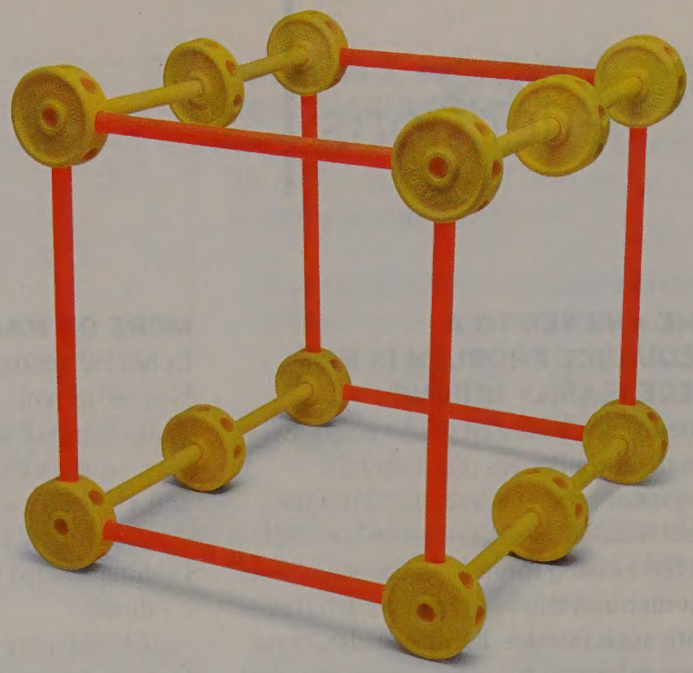

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- "A Random Walk: Stumbling across Connections," by Nicholas H. Wasserman
- "Mathematical Conversations to Transform Algebra Class," by Jennifer Earles Szydlik
- "Staircases, Towers, and Castles," Melike Kara, Cheryl L. Eames, Amanda L. Miller, and Annie Chieu

THE ANSWER TO A SEQUENCE PROBLEM IS NOT NECESSARILY UNIQUE

Intelligence tests and works of mathematical challenges often contain sequence problems such as "Find the next number in the sequence" or "Fill in the missing number in the sequence." I sometimes snicker when confronted with such puzzles, because a classmate once informed me, many years ago, that by replacing the expected sequence-generating function with your own, you may render correct any preselected number.

For example, we presume that the next number in the sequence {6, 12, 18, . . .} is 24. That is, we anticipate that the rule for the nth term is $6n$. On the other hand, if we want the answer to be 30, we can fit a cubic to the points (1, 6), (2, 12), (3, 18), and (4, 30). Then the nth term is $a_n = n^3 - 6n^2 + 17n - 6$. (Incidentally, I'm grateful to technology for replacing the formerly lengthy process of finding a fit with the routine one of data entry.)

Similarly, the missing third number in the sequence $\{1, 8, a_3, 64, \ldots\}$ is meant to be 27. That is, we expect the nth term to be n^3. But if we want a_3 to be 9, then we fit a cubic to (1, 1), (2, 8), (3, 9), and (4, 64). Accordingly, we adopt our own rule: $a_n = 10n^3 - 63n^2 + 126n - 72$.

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Lewiston, NY, March 29, 2014

We appreciate the interest and value the views of those who write. Readers commenting on articles are encouraged to send copies of their correspondence to the authors. For publication: All letters for publication are acknowledged. Letters to be considered for publication should be in MS Word document format and sent to **mt@nctm.org**. Letters should not exceed 250 words and are subject to abridgment. At the end of the letter include your name and affiliation, if any, including email address, per the style of the section.

MORE ON MAGIC NUMBERS

In her reflection "Magic Numbers" (*MT* Nov. 2014, vol. 108, no. 4, pp. 248–49), Anita Schuloff described "magic numbers," c, such that both $x^2 + bx + c = 0$ and $x^2 + bx - c = 0$ have integer solutions. Schuloff wondered whether a formula existed for generating these c-values.

If p and q are roots of $x^2 + bx + c = 0$, then $c = p \cdot q$ and $b = -(p + q)$. For $x^2 + bx - c = 0$, the discriminant would be $b^2 + 4c = (-(p + q))^2 + 4pq = p^2 + 6pq + q^2$; its roots would be

$$\frac{\left(-b \pm \sqrt{b^2 + 4c}\right)}{2} = \frac{\left(p + q\right) \pm \sqrt{p^2 + 6pq + q^2}}{2}.$$

I posit that if $p^2 + 6pq + q^2$ is a perfect square, then pq and $-pq$ are magic numbers. It can be shown that if $p^2 + 6pq + q^2$ is a perfect square, then both values of

$$\frac{\left(p + q\right) \pm \sqrt{p^2 + 6pq + q^2}}{2}$$

are integers.

I wrote some BASIC code to test prospective values of c in order of increasing absolute values to test whether they were magic numbers. The program tested all possible pairs of factors of c (e.g., if $c = 6$, then the pairs are 1 and 6, –1 and –6, 2 and 3, –2 and –3) and consequent values of b. When $b^2 + 4c = p^2 + 6pq + q^2$ was a perfect square, I printed the values shown in **figure 1 (Smith)**; these show the magic numbers in increasing order of absolute value.

Note that the roots of $x^2 + bx + c = 0$ and $x^2 + bx - c = 0$ come in pairs and that they and their negatives form the same two magic numbers, c and $-c$. We leave it to readers to establish that one of these pairs has p and q both positive.

Therefore, one way of generating magic numbers is by programming nested loops for positive p and q. When $p^2 + 6pq + q^2$ is a perfect square, $c = pq$ is a magic number. **Figure 2 (Smith)** shows BASIC code to produce magic numbers. Although they are not generated in ascending order of c (the code to sort the list would distract from the algorithm, so it was omitted), no magic numbers less than 10,000 are missed.

Schuloff said that her students found multiples of earlier c values and associated roots for $c > 10{,}000$. I found multiples even with relatively small c; see the two columns in **figure 1 (Smith)** labeled "unique?" If p and q are

	|--------x^2+bx+c=0-------|					|--------x^2+bx-c=0-------|			
#	b	c	p	q	unique?	b	-c	r1	r2	unique?
-	-	-	-	-	-------	-	--	--	--	-------
1	5	-6	1	-6	yes	5	6	-2	-3	yes
2	-5	-6	-1	6	-1*(#1)	-5	6	3	2	-1*(#1)
3	-5	6	2	3	-1*(#1)	-5	-6	6	-1	-1*(#1)
4	5	6	-2	-3	1*(#1)	5	-6	1	-6	1*(#1)
5	10	-24	2	-12	2*(#1)	10	24	-4	-6	2*(#1)
6	-10	-24	-2	12	-2*(#1)	-10	24	6	4	-2*(#1)
7	-10	24	4	6	-2*(#1)	-10	-24	12	-2	-2*(#1)
8	10	24	-4	-6	2*(#1)	10	-24	2	-12	2*(#1)
9	13	-30	2	-15	yes	13	30	-3	-10	yes
10	-13	-30	-2	15	-1*(#9)	-13	30	10	3	-1*(#9)
11	-13	30	3	10	-1*(#9)	-13	-30	15	-2	-1*(#9)
12	13	30	-3	-10	1*(#9)	13	-30	2	-15	1*(#9)
13	15	-54	3	-18	3*(#1)	15	54	-6	-9	3*(#1)
14	-15	-54	-3	18	-3*(#1)	-15	54	9	6	-3*(#1)
15	-15	54	6	9	-3*(#1)	-15	-54	18	-3	-3*(#1)
16	15	54	-6	-9	3*(#1)	15	-54	3	-18	3*(#1)

Fig. 1 (Smith)

```
maxc=10000
print "|--------------------x^2+bx+c=0------------------|";
print "|--------------------x^2+bx-c=0------------------|"
print "b","c","p","q","b","-c","r1","r2"
print "-","-","-","-","-","-","-","-"
for p=1 to maxc
    for q=p+1 to maxc/p
        n=p*p+6*p*q+q*q
        x=int(sqr(n+.1))
        if x*x=n then
            b=-1*(p+q)
            c=p*q
            r1=(-1*b+x)/2
            r2=(-1*b-x)/2
            print b,c,p,q,b,-1*c,r1,r2
        end if
    next q
next p
```

Fig. 2 (Smith)

relatively prime, then c and associated roots are not multiples of any other smaller values. If p's value is replaced with kp and q's value with kq, then $c = (kp)(kq) = k^2pq$. My algorithm finds unique magic numbers in the range of c above 10,000 as well: 10374, 10626, 10710, and so on. (Extended tables are available at **www.nctm.mt071**.)

Scott Smith
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Cleveland, OH, Nov. 14, 2014

COUNTING ON FINGERS

I have a question for *Mathematics Teacher* readers: Has anyone noticed that many students these days are counting on their fingers to add and multiply? As a teacher who retired in 2005, after teaching mathematics and statistics in grades 7–14 for a total of thirty-five years, and who now volunteers as a tutor in a local grade 7 mathematics class, I have observed an increasing number of students counting on their fingers to get simple number facts, a situation that was and is handicapping them in progressing smoothly to the deeper work of which I have considered them capable.

To be fair, the Common Core State Standards Initiative (CCSSI 2000) document does stipulate that students will be able to "fluently add and subtract within 20 using mental strategies" in grade 2 (p. 19) and to "fluently multiply and divide within 100" in grade 3 (p. 23). This leads me to wonder whether memorization, as most rote work in all subjects, is being deemphasized on principle or whether the demands of increased content coverage have, in practice, taken classroom time away from it. I also don't know whether technological tools are supposed to eliminate the need for immediate recall of number facts.

A CCSSI document states that "the traditional U.S. mathematics curriculum needed to become substantially more coherent and focused in order to improve student achievement, addressing the problem of a curriculum that is 'a mile wide and an inch deep.'" It seems to me that we cannot go deeper without basic skills. One (anonymous) blogger puts it well:

> Beyond memorization or even fluency, important patterns become evident to those who are conversant at basic operations and perform those operations regularly. It goes back to learning the basics early and effectively: students who are fluent with their addition and multiplication tables are more likely to be able to answer abstract questions . . . because they won't be mired in the arithmetic.(http://ccssimath.blogspot.com/2014/08/dont-punt-it-away-take-skills-into-end.html)

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PROBLEM 3, SEPTEMBER 2014 CALENDAR

Problem 3 of the September 2014 Calendar (*MT* Sept. 2014, vol. 108, no. 2, p. 120) was stated as follows:

> The side lengths of a triangle are the integral roots of the equation
>
> $$x^3 - 16x^2 + 85x - 150 = 0.$$
>
> What is the area of the triangle?

For the given equation having roots a, b, and c, "theorems" commonly attributed to François Viète and later to Isaac Newton provide the following:

$$a + b + c = 16$$
$$ab + ac + bc = 85$$
$$abc = 150$$

We thereby note that the sum of the roots—that is, $a + b + c = 16$—corresponds precisely to the perimeter of the triangle. This brings to mind Heron's formula for the area of a triangle—namely,

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where A is the area of the triangle; a, b, and c are its side lengths; and $s = (a + b + c)/2$ is its semiperimeter. If we partially expand the formula, we are led to

$$A = \sqrt{s\left(-abc + s\left(ab + ac + bc\right) - s^2\left(a + b + c\right) + s^3\right)},$$

at which point we recognize the quantities found earlier using "Viète's" relationships between roots and coefficients. Substituting into the expanded form of Heron's formula, we have

$$A = \sqrt{8\left(-150 + 8\left(85\right) - 8^2\left(16\right) + 8^3\right)} = \sqrt{144} = 12.$$

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Wayzata, MN, Sept. 8, 2014

Calendar editors' note:
If the side lengths of a triangle are the zeros of $f(x) = x^3 - px^2 + qx - r$ (so that p

is the sum of the roots), then the area of the triangle is

$$\sqrt{(p/2)\cdot f(p/2)}.$$

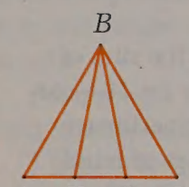
PROBLEMS 15 AND 20, SEPTEMBER 2014 CALENDAR

These two problems have alternate paths to the solutions.

Problem 15

Problem 15 (*MT* Sept. 2014, vol. 108, no. 2, p. 120) was stated as follows:

> Equilateral $\triangle ABC$ has side length 12. Points X and Y trisect $\overline{AC}$, so triangles ABX, XBY, and YBC have equal area. But their perimeters are not equal. If $a - b\sqrt{c}$, written in simplified radical form, is the positive difference between the perimeter of $\triangle ABX$ and $\triangle XBY$, what is the value of $a+b+c$?



An alternative solution uses the law of cosines:

$$BX^2 = BY^2 = 12^2 + 4^2 - 2(12)(4)\cos 60°$$
$$= 112$$
$$BX = BY = 4\sqrt{7}$$

Then the perimeter of $\triangle ABX$ minus the perimeter of $\triangle XBY$ equals $(12 + 4 + 4\sqrt{7}) - (4 + 8\sqrt{7}) = 12 - 4\sqrt{7}$. Hence, $a = 12$, $b = 4$, and $c = 7$. Thus, $a + b + c = 23$.

Problem 20

Problem 20 (*MT* Sept. 2014, vol. 108, no. 2, p. 121) was stated as follows:

> A circle with radius $\sqrt{170}$ passes through the points (0, 2) and (2, 8). If the center of the circle lies in quadrant I, find the x-coordinate of the center of the circle.

Let the center of the circle be (h, k) so that the equation of the circle is $(x - h)^2 + (y - k)^2 = 170$. Because (0, 2) lies on the circle, $h^2 + (2 - k)^2 = 170$. Similarly, (2, 8) lies on the circle; hence,

$$(2-h)^2 + (8-k)^2 = 170$$
$$(2-h)^2 + \left[6 + (2-k)\right]^2 = 170$$
$$(2-h)^2 + 36 + 12(2-k) + (2-k)^2 = 170$$
$$4 - 4h + h^2 + 36 + 12(2-k) + 170 - h^2 = 170$$
$$12(2-k) = 4h - 40 \text{ or}$$
$$3(2-k) = h - 10.$$

On squaring, we have

$$9(2-k)^2 = h^2 - 20h + 100$$
$$9(170 - h^2) = h^2 - 20h + 100$$
$$10h^2 - 20h - 1430 = 0 \text{ or}$$
$$h^2 - 2h - 143 = 0 = (h-13)(h+11).$$

But the center is in the first quadrant; hence, $h = 13$.

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MORE ON PROOF EXAMPLES

Kudos to Garofalo, Trinter, and Swartz for their article "Engaging with Constructive and Nonconstructive Proof" (*MT* Feb. 2015, vol. 108, no. 6, pp. 422–28). Regarding task 1, readers may find additional illumination from the representations of Simpson's paradox explored with preservice secondary school teachers in my 2001 NCTM Yearbook chapter ("Representations of Reversal: An Exploration of Simpson's Paradox," pp. 129–45; available with permission at http://www.statlit.org/pdf/2001LesserNCTM.pdf).

Regarding task 3, readers may be interested to know that we actually can say which example is correct. Recall that an algebraic number is a solution to a polynomial equation (then $\sqrt{2}$ is algebraic because it satisfies $x^2 - 2 = 0$) so that all rational numbers are algebraic, but not all algebraic numbers are rational. By the Gelfond-Schneider theorem, if a is an algebraic number other than 0 or 1 and b is an irrational algebraic number, then a^b is transcendental (i.e., not algebraic, implying that it cannot be rational). Thus, we can conclude that $\sqrt{2}^{\sqrt{2}}$ itself is transcendental and will yield a number that is rational when it is raised to the power $\sqrt{2}$.

A constructive example of an irrational number raised to the power of an irrational (though not algebraic) number that yields a rational number that does not involve only radicals and exponents and yet is accessible to high school students is $\sqrt{2}^{\log_2 9} = 2^{\log_2 3} = 3$. Finally, it is worth adding that when calculus classes apply the intermediate value theorem or mean value theorem, they are asserting the existence of a value without stating it or necessarily having a way to find it.

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MEDIA clips

Michael A. Jones and Jennifer M. Wilson

Playing Winless Teams

As the great Joe Posnanski pointed out this morning on his blog, when the Redskins host the 0-5 Kansas City Chiefs this Sunday, it will mark the team's sixth consecutive game against a winless team. I'll repeat that because it doesn't sound as impressive the first time around: So far this season, every team the Redskins have played has been winless at the time of the game. According to STATS LLC, it's the first time in the history of the NFL that this has happened. The last team to have faced five winless teams was the 1954 Giants.

According to . . . my dad (whom I had to call to get help with the equation), the odds of this happening are 1 in 32,768. (Feel free to check this and let me know if our numbers are wrong.)

Here's the way the Redskins schedule has broken down so far, along with their opponent's record at the time:

Week 1 — at New York Giants (0-0)
Week 2 — vs. St. Louis Rams (0-1)
Week 3 — at Detroit Lions (0-2)
Week 4 — vs. Tampa Bay Buccaneers (0-3)
Week 5 — at Carolina Panthers (0-3 — *Note: Carolina had a week 3 bye*)
Week 6 — vs. Kansas City Chiefs (0-5) . . . .

Source: Chris Chase, "Redskins Are First in History to Play Six Straight Winless Teams," posted October 12, 2009, on Shutdown Corner, a Yahoo! Sports Blog at http://sports.yahoo.com/nfl/blog/shutdown_corner/post/Redskins-are-first-in-history-to-play-six-straig?urn=nfl,195479

For the following questions, we will make these assumptions:

- Each team wins or loses each game with a likelihood of 1/2.
- Each game is independent of other games.

1. What is the probability that the Detroit Lions were winless in week 3, when they faced the Washington team?

2. What is the probability that Washington's first 6 opponents were winless? How does this compare with the 1 in 32,768 probability mentioned in the clip?

3. How likely would it be for Washington to win its first 6 games? How likely would it be for the team to win its first 6 games if it played undefeated teams? How do your answers relate to the clip's implication that Washington had an easy schedule because it faced only winless teams in its first 6 games?

4. To dissuade teams from resting too many players in preparation for the playoffs, the National Football League instituted a policy to have division rivals play one another in the last game of the season. The Washington team is part of the NFC East Division along with the Dallas Cowboys, New York Giants, and Philadelphia Eagles. Each team in the division must play the other teams twice during the season. If we assume that the Washington team plays the Cowboys and Giants for its last two games of the season, is it possible for the team to play all 16 games against winless teams?

Media Clips appears in every issue of *Mathematics Teacher*, offering readers contemporary, authentic applications of quantitative reasoning based on print or electronic media. All submissions should be sent to the editors.

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Ranking the Safest U.S. Cities

To determine our list of America's safest cities, we looked at the country's 40 largest metropolitan statistical areas across four categories of danger. We considered violent crime rates from the 2008 Federal Bureau of Investigation (FBI) Uniform Crime Report; 2008 workplace death rates from the Bureau of Labor Statistics; 2008 traffic death rates from the National Highway Traffic Safety Administration; and natural disaster risk, using rankings from green living site SustainLane.com. . . .

Rank	Metro Area	MSA Population	Violent Crime	Workplace Fatality Rates	Traffic Death Rates	National Disaster Risk	Total
1	Minneapolis–St. Paul–Bloomington, MN–WI	3,229,878	9	1	7	7	24
2	Milwaukee–Waukesha–West Allis, WI	1,549,308	24	11	4	1	40
3	Portland–Vancouver–Beaverton, OR–WA	2,207,462	1	10	5	25	41
4	Boston–Cambridge–Quincy, MA–NH	4,522,858	10	5	1	28	44
4	Seattle–Tacoma–Bellevue, WA	3,344,813	3	2	8	31	44
6	Providence–New Bedford–Fall River, RI–MA	1,596,611	6	7	11	28	52
7	San Jose–Sunnyvale–Santa Clara, CA	1,819,198	2	8	6	38	54
8	New York–Northern New Jersey–Long Island, NY–NJ–PA	19,006,798	11	15	2	27	55
9	Cincinnati–Middletown, OH–KY–IN	2,155,137	8	27	20	2	57
10	Cleveland–Elyria–Mentor, OH	2,088,291	15	25	17	2	59
10	Denver–Aurora, CO	2,506,626	5	28	16	10	59
12	Detroit–Warren–Livonia, MI	4,425,110	40	4	10	8	62
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Source: Zack O'Malley Greenburg, "America's Safest Cities," *Forbes Magazine*, October 26, 2009, http://www.forbes.com/2009/10/26/safest-cities-ten-lifestyle-real-estate-metros-msa_chart.html

In this clip, each city was given a ranking (an integer from 1 to 40) in each of the four categories, and then the rankings were summed to arrive at the overall ranking. The safest city had the smallest sum, the second safest city had the second smallest sum, and so on. A complete table showing all forty cities is available online as an Excel file (go to **www.nctm.org/mt072**).

1. Explain how you know that the data used to rank the cities in each category took into account the population of each city (meaning that the data are *per capita*). How would the rankings change if the data were not per capita?

2. Rankings over the four criteria are ordinal numbers; thus, the numbers do not have any units. The "total" column is derived by adding these numbers from each of the four criteria. What assumptions are being made about the data when these four numbers are added to produce an overall ranking?

3. The four criteria in the table in the clip are treated as equal in importance. Suppose that you decide to weigh the criteria differently so that, compared with the natural disaster risk, violent crime is weighed four times as much, workplace fatality rates are weighed twice as much, and traffic death rates are weighed three times as much. Then, for example, New York's score becomes

$$(4 \cdot 11) + (2 \cdot 15) + (3 \cdot 2) + (1 \cdot 27) = 107.$$

   Using this weighting, referred to as (4, 2, 3, 1), and the full ranking shown in the Excel file online, find which city is ranked the safest.

4. The Boston metropolitan area is ranked first in traffic death rates. Explain how to adjust the weights so that Boston is considered the all-around safest city.

5. A New York Yankees baseball fan wants to adjust the weights used on

the criteria in such a way that the New York metropolitan area ranks above Boston in all-around safety. Is this possible? If so, provide weights to accomplish the goal. If not, explain why not.

6. A Detroit Lions fan decides to do something similar—adjust the weights so that the Minneapolis metropolitan area (home of the Vikings football team and currently the safest city in the study) ranks below Detroit. Is this possible? If so, provide weights to accomplish the goal. If not, explain why not.

7. For violent crime, workplace fatality rates, and traffic death rates, the data show no ties; each integer 1–40 appears once in each column. However, for natural disaster risk, ties are possible. In fact, the full data set shows a three-way tie for 2nd place, a tie for 16th, a three-way tie for 22nd, a tie for 28th, and a tie for 35th. If these rankings are treated as points, then the two metropolitan areas tied for 35th place—Los Angeles and Riverside, California—should each receive $(35 + 36)/2 = 35.5$ points for taking up positions 35 and 36 in the list. If we allow for ties to be evaluated this way, how would the natural disaster risk and overall rankings for all the cities change in comparison with the original report?

8. The violent crime ranking comes from the 2008 FBI Uniform Crime Report (http://www.fbi.gov/stats-services/crimestats). The link to the report includes a pop-up warning: "Each year when *Crime in the United States* is published, some entities use reported figures to compile rankings of cities and counties. These rough rankings . . . lead to simplistic or incomplete analyses that often create misleading perceptions adversely affecting communities and their residents." Is such a warning warranted for the overall safety rankings?

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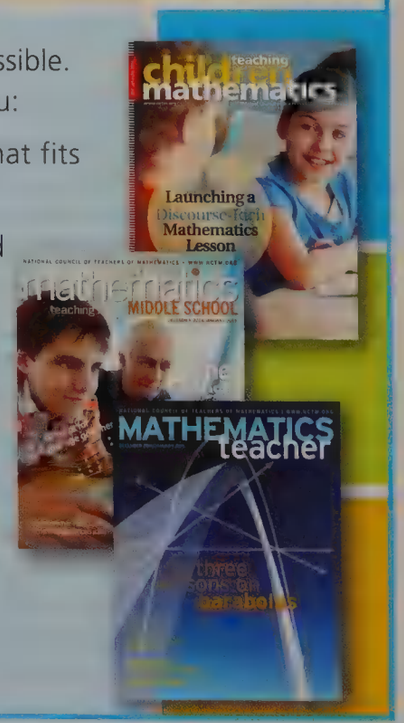

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MEDIA CLIPS solutions

"Playing Winless Teams" answers

1. The Lions had played 2 games before facing the Washington team. Because we have assumed that the results of each game are independent of one another and the probability of losing any game is 1/2, the likelihood that the Lions had lost both games before playing Washington is $(1/2) \cdot (1/2) = (1/2)^2$, or 1/4.

2. The probability that the St. Louis Rams were winless when they played Washington is (1/2); the probability that the Tampa Bay Buccaneers were winless when they played Washington is $(1/2)^3$; and so on. Since each of these probabilities is independent, the probability that Washington's first 6 opponents were winless is

$$(1/2) \cdot (1/2)^2 \cdot (1/2)^3 \cdot (1/2)^3 \cdot (1/2)^5 = (1/2)^{14} = 1/16384.$$

   The probability given in the article is half this value, or 1/32768. The blogger's father may not have been aware that the Carolina Panthers had a bye in week 3 and may have used $(1/2)^4$ for the probability that the Panthers lost 4 games before playing Washington.

3. The probability that Washington won its first 6 games is $(1/2)^6 = 1/64$. If Washington had faced only teams with perfect records, the probability that it won its first 6 games would still be $(1/2)^6 = 1/64$. As sports fans, we know that the assumption that any given team wins with equal probability is likely not the case. To investigate the claim that Washington's schedule was easy, we would need far more information to determine the probability of any team winning.

4. The Dallas Cowboys and the New York Giants would have played each other twice before the last two games that Washington played. Thus, assuming no ties, one of the two teams must have won a game. Under such a schedule, it would be impossible for Washington to play a full season against winless teams. (Since a rule change in 2011, only three ties have occurred in the NFL.)

"Ranking the Safest U.S. Cities" answers

1. The word *rate* implies a relative measure. It is likely that the number of deaths in a city is proportional to the number of inhabitants of that city. If the rankings did not account for per capita measures, the largest cities would naturally be at the bottom of each category. (See the complete table at **www.nctm.org/mt072**.)

2. Numbers across categories may have different meanings. A rank of 1 in workplace fatality rates likely does not mean the same number of deaths per capita as a rank of 1 in traffic death rates. Increments within categories may have different meanings. An increase from 2 to 5 in violent crime may not mean the same thing as an increase from 12 to 15. Producing an overall ranking by adding the numbers assumes that values are comparable within and between categories. Also other influential factors that might relate to safety, such as accessibility to quality health care, transportation options, and poverty levels, are not considered.

3. If we use the weighting (4, 2, 3, 1), Portland–Vancouver–Beaverton, OR–WA would be ranked the safest city or metropolitan area, with a score of

$$(4 \cdot 1) + (2 \cdot 1) + (3 \cdot 5) + (1 \cdot 25) = 64.$$

4. The traffic death rates data can be weighted so that they "weigh down" the other metropolitan areas, allowing Boston to be ranked first. One way to do this uses the weighting (1, 1, 5, 1); Boston is then in 1st place, with a score of 48.

5. Because New York is ranked above Boston for natural disaster risk, it is possible to adjust the weights so that New York is ranked above Boston. Raising the weight on natural disaster risk by using, say, (1, 1, 1, 13), gives Boston 380 points and New York 379 points. A downside is that New York's rank would fall to the 27th safest city.

6. Because Minneapolis is ranked higher than Detroit in all four categories, it is impossible to adjust the positive weights in such a way that Detroit is ranked above Minneapolis in any aggregate ranking.

7. The following changes would have to be made to the natural disaster risk ranking: The three cities tied for 2nd place would each receive 3 points; the two cities tied for 16th would each receive 16.5 points; the three cities tied for 22nd would each receive 23 points; and the two cities tied for 28th would each receive 28.5 points. The effect would be that Boston would no longer be tied for 4th place overall but would be in 5th place, whereas Cleveland would no longer be tied for 10th place but would be in 11th place.

8. Yes, this warning is necessary, especially given the answer to question 2. Also, as demonstrated, if some criteria are viewed as more important than others, then the overall ranking can change dramatically. Because metropolitan areas cover vast geographic areas, which are not homogeneous, a single number may be inadequate to characterize the entire region. Further, a person's individual safety depends on a vast number of variables, including gender, age, race, social class, neighborhood, job, relationships, and work environment. All these criteria should be considered for the data rankings to be meaningful.

For the complete table of all forty cities in Excel file format ("Ranking the Safest U.S. Cities"), download one of the free apps for your smartphone and then scan this tag to access **www.nctm.org/mt072**.

Douglas Wilcock

Shine On, Shine Dome

Shine Dome (**photographs 1** and **2**), named for the Australian biologist John Shine, a major contributor to the building's renovation, is a striking landmark in the Australian capital city of Canberra. Constructed in 1959 as the home of the Australian Academy of Science, the dome is a mathematically interesting structure featuring arches and a surrounding moat. The dome, a spherical cap, is supported by concrete piers grounded in the moat.

The most striking architectural features of Shine Dome are the sixteen arches at its base. They permit light, some of which is reflected from the water in the moat, to enter the building, and the net effect is a light, airy feel. The building's load is placed on sixteen piers located between the arches, thus giving a massive concrete structure an architectural grace. Further, the shadows cast by the arches add an additional pleasing aesthetic (see **photograph 3**).

Photograph 1
Shine Dome, Canberra, Australia

Photograph 2
Aerial View of Shine Dome

Photograph 3
Concrete piers intersecting ground level in the moat

Mathematical Lens uses photographs as a springboard for mathematical inquiry and appears in every issue of *Mathematics Teacher*. All submissions should be sent to the department editors.

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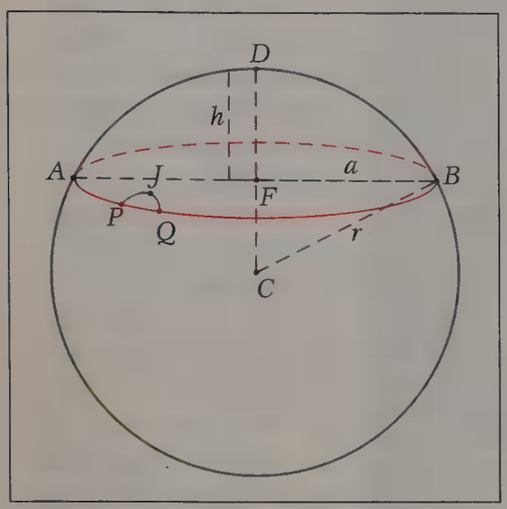

Fig. 1 A spherical cap represents the dome with its circular base through points *A* and *B*.

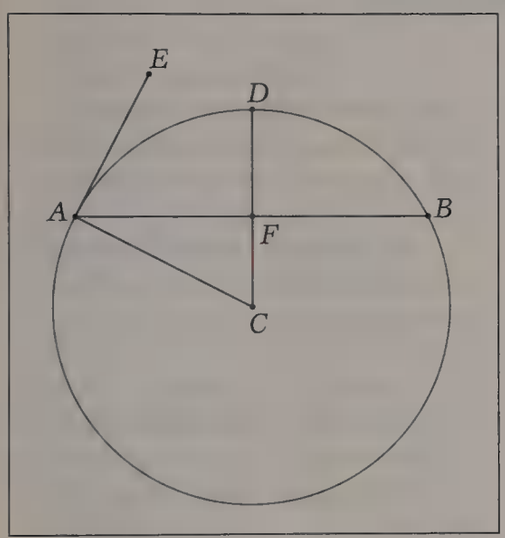

Fig. 2 A section shows *AE* as a tangent to the dome at ground level.

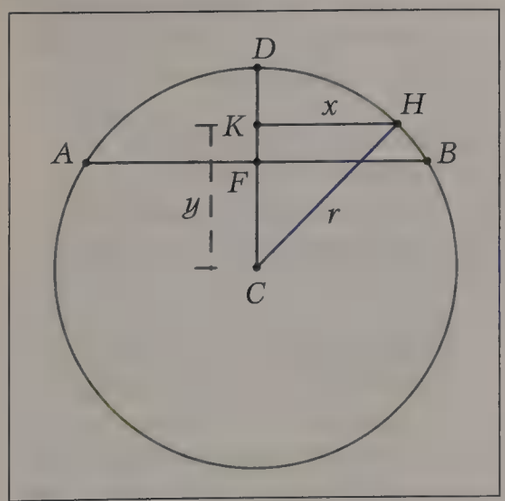

Fig. 3 Related lengths can be used to find the surface area of the dome.

1. Shine Dome is contained in a part of a sphere called a spherical cap (see **fig. 1**). If we cut through the dome perpendicularly to the circular floor, we get a section showing the height of the dome, *DF*, and the diameter of the circular floor, represented by segment *AB*. In **figure 2**, a tangent line to the dome at the ground level is labeled *AE*, and two radii of the sphere are *AC* and *CD*. The diameter of the dome, *AB*, is 45.75 meters.

   (*a*) Determine a formula for the height of the dome as a function of the radius of the sphere.

   (*b*) Restrict the height of the dome such that $DF \leq DC$ and modify your answer to part (*a*).

   (*c*) What is the height of the dome when the radius of the sphere is 22.875 m? How would you describe the shape of the dome?

2. The height of the dome represented in **figure 2** can be calculated as a function of the angle formed by the tangent line *AE* and the floor of the dome *AB*.

   (*a*) Determine the height of the dome if the measure of $\angle EAB$ is 30°, 45°, 60°, and 90°. List the results in **table 1**. (The radius of the sphere [see **table 1,** col. 2] will be an intermediate calculation in this process.)

   (*b*) Determine a general formula for the height of the dome as a function of $\theta = m\angle EAB$ and enter your formula in the last line of **table 1**. (Column 4 of **table 1** will be completed in question 4.)

   (*c*) If we restrict the domain of the function in part (*b*) to be positive values less than or equal to 90°, what is the range of the function?

3. (*a*) Refer to **figure 3**, a section of the dome, where segment *AB* represents the floor of the dome, and *FD* is the height of the dome. Use the center of the dome, *C*, and the (variable) point *K* on segment *FD* to define a point *H* (at the same height as *K* but on the surface of the dome) and variables $x = KH$ and $y = CK$ in right triangle *CHK*. Solve for *x* as a function of *y* and *r*, the radius of the sphere. Notice that *x* is the radius of a circular disk parallel to the floor at the same height as point *K*.

   (*b*) The formula for the (outside) surface area of the spherical cap is $SA = 2\pi rh$, where *r* is the radius of the sphere and *h* is the height of the dome. Develop this formula (using calculus).

4. Complete the last column in **table 1**, using the formula in question 3. That is, find the surface area of the dome for each angle listed.

Table 1 Dome Measurements for Questions

Tangent Angle $m\angle EAB$	Radius of Sphere $r = AC$ or DC (in m)	Height of Dome $h = DF$ (in m)	Gross Surface Area of Dome (in m^2)
30°			
45°			
60°			
90°			
θ			

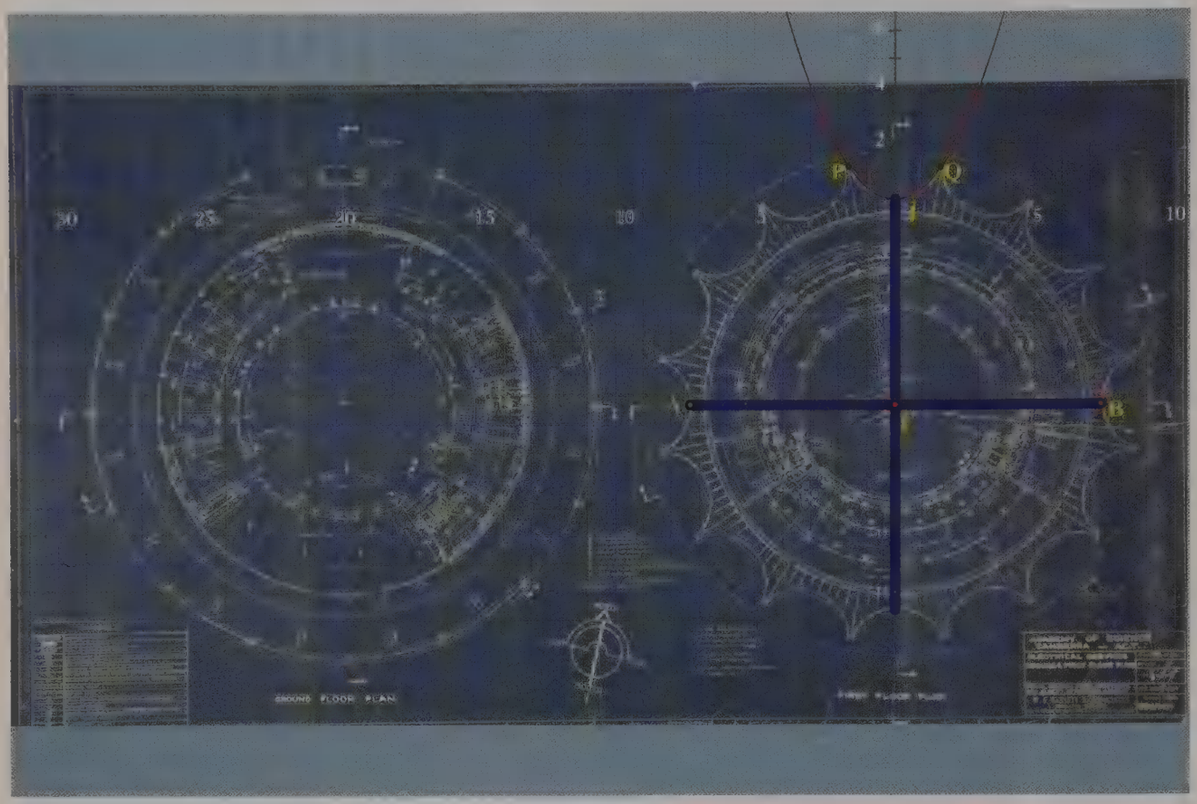

Fig. 4 A blueprint of dome shows a parabola representing one of the arches.

5. **Figure 4** shows a blueprint of the dome and a parabola superimposed over one of the arches. Point F is located at the center of the dome's floor; point J is a projection of the apex of the arch, or the vertex of the parabola, onto the floor; and point Q is located at the foot of the pier where the arch meets ground level. The diagram coordinates (in cm) of each point are F (0, −7.34), J (0, 0), and Q (1.68, 1.24). Point P is a reflection of point Q across the axis of symmetry of the parabola.

   (*a*) Segment FQ represents a radius of the dome's circular floor with an actual length of 22.875 m. Find the length (using coordinates in cm) of FQ on the blueprint and determine a scaling factor for the other coordinates and actual distances (in m).

   (*b*) Determine the area enclosed by one parabolic arch—that is, the region bounded by segment PQ and the parabola. The area of a parabolic region can be calculated with a formula as 2/3 of the area of the smallest rectangle containing the parabola or using calculus.

6. The answer that was determined in question 5 represents the area of a parabolic arch projected on the ground. However, the plane containing the dome's arch forms an angle with the ground. Assume that the plane of the parabolic arch forms a 45° angle with the ground and that the apex of the arch, G, is directly above the vertex of the parabola on the ground, J.

   (*a*) Determine the area of one arch in the tilted plane.

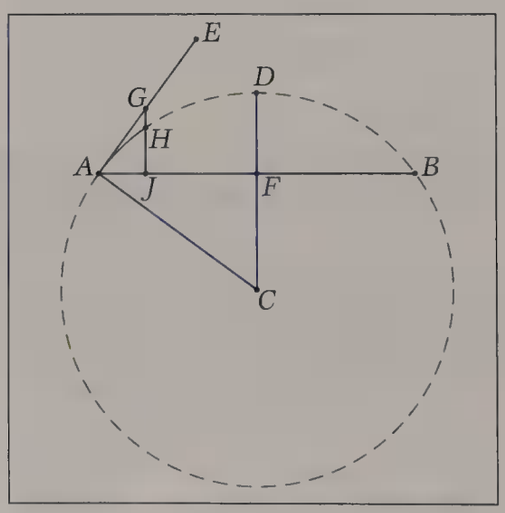

Fig. 5 The dome's arches could be represented by parabolas with vertices at G or H.

   (*b*) Determine the net surface area of the dome that results when 16 arches are removed.

   (*c*) Determine the dome's weight if its average thickness is 20 cm and the reinforced concrete weighs approximately 2500 kg/m^3. Convert the answer to metric tons (1 metric ton = 1000 kg).

   (*d*) At ground level, the piers are squares with side lengths of 0.6 m. Determine the load on each of the 16 piers in kg per pier and as metric tons per m^2.

7. The calculations in question 6 determine the area of a parabolic arch with an axis of symmetry represented in **figure 5** as AG. Because of the curvature of the dome, other approximations for the area would use the axis of symmetry of the parabola along segment or arc AH (see **fig. 5**).

   (*a*) Assume that, in **figure 5**, m$\angle EAB$ = 45° and find the position of H, relative to F, the point in the center of the dome's circular floor.

   (*b*) Use the results of your calculations in part (*a*) to determine the arc length of arc AH and the area of a parabolic arch with arc AH as its axis of symmetry.

   (*c*) Compare the parabolic-arch areas calculated with axis of symmetry AJ, AG, and AH. What are the safety implications for the engineers who designed the building?

   Readers might also consider another approximation that uses slices perpendicular to the dome's altitude so that the arch width at any height could be approximated as a circular arc length rather than the width of a parabola.

MATHEMATICAL LENS
solutions

1. (*a*) In **figure 2**, we solve for the dome's height, h, by using the sphere's radius, r, and the relationship among the sides in right triangle AFC:

$$r^2 = (r-h)^2 + 22.875^2$$
$$r^2 = r^2 - 2rh + h^2 + 22.875^2$$
$$0 = h^2 - 2rh + 22.875^2$$
$$h = r \pm \sqrt{r^2 - 22.875^2}$$

(*b*) The dome's height would be the difference (using subtraction):

$$h = r - \sqrt{r^2 - 22.875^2}$$

Addition would make the height greater than the radius and put the dome floor in the bottom half of the surrounding sphere.

(*c*) If the radius is 22.875 m, the dome's height simplifies to r, or 22.875 m, and the resulting shape is a hemisphere.

2. (*a*) See **table 2**.

(*b*) Because segments AE and CF are perpendicular to each other, the measure of $\angle ACF$ is θ, so $\sin\theta = 22.875/r$. We use an expression for $\cos\theta$ to solve for h:

$$\cos\theta = \frac{r-h}{r}$$
$$r\cos\theta = r - h$$
$$h = r - r\cos\theta$$
$$h = r(1-\cos\theta).$$

(*c*) The range of the function h is from 0 to 22.875 m, the latter representing a tangent angle of 90°.

3. (*a*) The relationship is given by the Pythagorean theorem. Solving for x, we get

$$x = \sqrt{r^2 - y^2}.$$

(*b*) The surface area can be found by integrating the circumference of stacked circular disks with varying radius x (where x is as a function of y found in part (*a*)) along sections of arc ds—that is, by applying the formula

$$SA = \int 2\pi \cdot x \cdot ds$$

with appropriate limits of integration. You may find s as a function of y from $s = r\theta = r\sin^{-1}(y/r)$, so

$$\frac{ds}{dy} = \frac{r}{\sqrt{r^2 - y^2}};$$

thus, the integral becomes

$$\int 2\pi x\, ds = \int 2\pi\sqrt{r^2 - y^2}\,\frac{r}{\sqrt{r^2 - y^2}}\,dy.$$

Alternatively, you may use the formula

$$ds = \sqrt{1 + (x')^2}\,dy.$$

Then, taking a derivative of x with respect to y, we have

$$x' = \frac{-y}{\sqrt{r^2 - y^2}}.$$

Thus,

$$ds = \sqrt{1 + (x')^2}\,dy$$
$$= \sqrt{1 + \frac{y^2}{r^2 - y^2}}\,dy$$
$$= \sqrt{\frac{r^2}{r^2 - y^2}}\,dy$$
$$= \frac{r}{\sqrt{r^2 - y^2}}\,dy,$$

which will lead to the same integral.

The limits of the integral will correspond to points F and D; thus, the surface area becomes

$$SA = \int_{s(F)}^{s(D)} 2\pi x\, ds$$
$$= \int_{y(F)}^{y(D)} 2\pi\sqrt{r^2 - y^2}\,\frac{r}{\sqrt{r^2 - y^2}}\,dy$$
$$= \int_{r-h}^{r} 2\pi r\, dy$$
$$= 2\pi r\big(r - (r-h)\big)$$
$$= 2\pi rh.$$

4. See the results in **table 2**.

5. (*a*) In the blueprint (see **fig. 4**), FP and FQ are the radii of the circular floor of the dome measuring 22.875 m in actual length. To find the scaling factor, we first calculate the blueprint length of FQ:

Table 2 Dome Measurements Depending on Tangent Angle at the Base

Tangent Angle m∠*EAB*	Radius of Sphere *r* = *AC* or *DC* (in m)	Height of Dome *h* = *DF* (in m)	Gross Surface Area of Dome (in m²)
30°	45.75	6.13	1761.91
45°	32.35	9.48	1925.93
60°	26.41	13.21	2191.85
90°	22.875	22.875	3287.77
θ	$r = 22.875/\sin\theta$	$h = r(1-\cos\theta)$	

$$FQ = \sqrt{1.68^2 + (1.24 - (-7.34))^2} \approx 8.743$$

Then we use

$$\frac{22.875 \text{ actual length in m}}{8.74 \text{ blueprint length in cm}} \approx 2.617.$$

(Values are rounded to answer the questions, but nonrounded values are used at the intermediate calculations.)

(*b*) The distance from the vertex of the parabola, point *J*, to line segment *PQ* measures 1.24 cm in the blueprint (see **fig. 4**) and represents an actual distance of $1.24 \cdot 2.617 \approx 3.244$ m. Segment *PQ* represents an actual distance of $(1.68 - (-1.68)) \cdot 2.617 \approx 8.791$ m. The product of these lengths multiplied by 2/3 gives a parabolic-arch area of 19.014 m^2.

Using calculus, we can evaluate an integral to determine the area. Because *Q* lies on a parabola with vertex (0, 0), its coordinates are scaled from (1.68,1.24) in cm on the blueprint to a corresponding position (4.396, 3.244) in m on the parabola with an equation

$$y = 0.1679x^2 \text{ or } x = \sqrt{5.955y}.$$

Integrating with respect to *y*, we determine the area of the parabolic section to be

$$Area = 2\int_0^{3.244} \sqrt{5.955y}\, dy \approx 19.014.$$

6. (*a*) If the parabola is tilted at an angle of 45° to the ground and stretched so that the apex of the arch, *G*, is directly above point *J* on the ground, then the effective height of the parabolic section is multiplied by $\sqrt{2}$. That is, $AG \approx 4.588$ m. Because the base, *PQ*, of the section is unchanged, the area of slanted arch is $\sqrt{2}$ times 19.014, or approximately 26.89 m^2.

(*b*) In question 3 (*b*), we determined that the surface area of the spherical cap is 1925.93 m^2. We now remove 26.89 m^2 for each of the 16 arches, or 430.24 m^2 total. The net surface area for the dome is approximately 1495.69 m^2.

(*c*) The volume of the dome is approximately $1495.69 \cdot 0.2 = 299.14$ m^3. This volume gives a weight of approximately $282.857 \cdot 2500 = 747845$ kg = 747.845 metric tons.

(*d*) This weight distributes as $747845/16 \approx 46740$ kg per pier, or $46740/0.36 \approx 129834$ kg per $m^2 \approx$ 129.8 metric tons per m^2. This is a tremendous load, so it is not surprising to learn that a scale model (1/40 scale) was built before construction of Shine Dome to determine whether the piers could carry the load.

7. (*a*) *A* and *H* are points on the circle centered at *C* with radius $22.875 \cdot \sqrt{2} \approx 32.35$ m. We know that the distance $CF = 22.875$ m, and in the solution to question 5(*b*), we determined that the length *AJ* is 3.244 m. Thus, $FJ = 22.875 - 3.244 = 19.631$ m. It follows that vertical distance from *C* to *H* is

$$\sqrt{32.35^2 - 19.63^2} \approx 25.71 \text{ m},$$

so the point *H* is 25.71 – 22.875 = 2.84 m above the floor.

(*b*) To determine arc length *AH*, use the *x*-coordinates (relative to point *C*) for points *H* and *A* as –19.63 and –22.875, respectively. Then, points *H* and *A* correspond to rotations of $\cos^{-1}(-19.63/32.35) \approx 127.36°$ and 135°, respectively. Thus, $m\angle ACH \approx 7.64°$, and

$$\text{arc } AH \approx \frac{7.64}{360} \cdot 2\pi \cdot 22.875\sqrt{2} \approx 4.314.$$

The area of the arch, then, is $(2/3) \cdot 8.791 \cdot 4.314 \approx 25.283$ m^2.

(*c*) In calculating the parabolic arch areas as 2/3 the product of a base length *PQ* times a height, we found the heights $AJ \approx 3.244$, $AG \approx 4.588$, and arc $AH \approx 4.314$. Had the engineers used the area in the 45° plane (with parabola vertex at *G*), they would have underestimated the actual load on the piers, a dangerous simplifying assumption.

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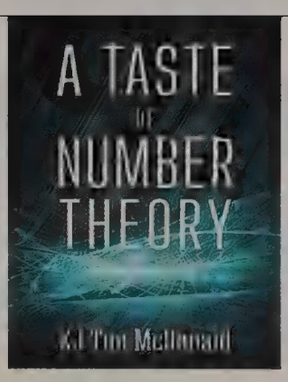

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DOUGLAS WILCOCK, dougw_96@comcast.net, retired as a mathematics, statistics, and economics teacher at Cape Cod Academy in Osterville, Massachusetts. He remains interested in finding mathematical applications in the world around us.

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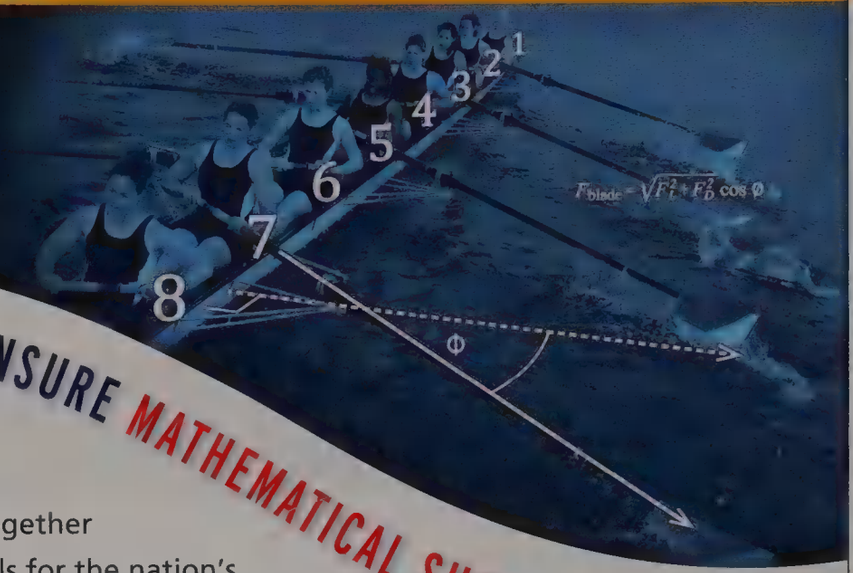

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Textbooks offer some opportunities to practice modeling, but we can do more for students to experience where mathematics and science meet.

MISSING THE Math

Dan Meyer

The Common Core State Standards for Mathematics (CCSSM) have exerted enormous pressure on every participant in a child's education. Students are struggling to meet new standards for mathematics learning, and parents are struggling to understand how to help them. Teachers are growing in their capacity to develop new mathematical competencies, and administrators are growing in their capacity to support them.

These standards have also exerted pressure on textbook publishers, who must provide curriculum that aligns with the CCSSM. The CCSSM have made some of this existing content obsolete or pushed it to other grade levels. In other cases, publishers have had to develop new content aligned to the CCSSM. But a recent study of fourth-grade textbooks found that this alignment has been slippery, with many textbooks including content external to the CCSSM, failing to include critical CCSSM content or duplicating their previous unaligned editions to an inappropriate degree (Polikoff 2014).

This situation should concern us all given the large sums of money spent nationally on textbooks and the high degree to which teachers take their instructional cues from textbooks. What incentives do publishers have to undertake these costly alignments and developments? The CCSS issued a Publishers' Criteria, but these criteria are not binding

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PROMISE OF
ematical Modeling

in any sense. We—the people who buy textbooks or influence those who do—are publishers' only incentive.

With that rationale in mind, what follows is my analysis of how well textbooks fulfill the promise of one particular standard—mathematical modeling—as it is represented in the CCSSM.

I choose to examine modeling for several reasons. First, speaking strictly personally, I studied mathematics as a child and mathematics education as an adult because of powerful experiences I had using mathematics as a model for the world around me. I want students to have similar experiences. Second, in my work with teachers in professional development, I find modeling with mathematics (Standard for Mathematical Practice 4, CCSSI 2010, p. 7) to be one of the practice standards most in need of explication. Five different teachers may have five different understandings of its meaning. Third, mathematical modeling is the standard where mathematics and science meet. The practice standards of the Next Generation Science Standards (NSTA 2012) resemble the mathematical modeling standards of the CCSSM so closely that we should ensure that we get our end right.

In my research, I analyzed two textbooks in particular—an algebra 1 textbook and a geometry textbook, both published by McGraw-Hill (Carter et al. 2013a, 2013b). I chose these particular texts

because, whereas other publishers might simply identify a general CCSSM alignment in their textbooks, these two textbooks made the specific claim that their tasks captured specific standards such as "modeling." It seemed fair then to ask, "How well?" Through my analysis, I identified eighty-three tasks across the two years of mathematics that claimed to represent "modeling with mathematics."

MODELING IN THE CCSSM

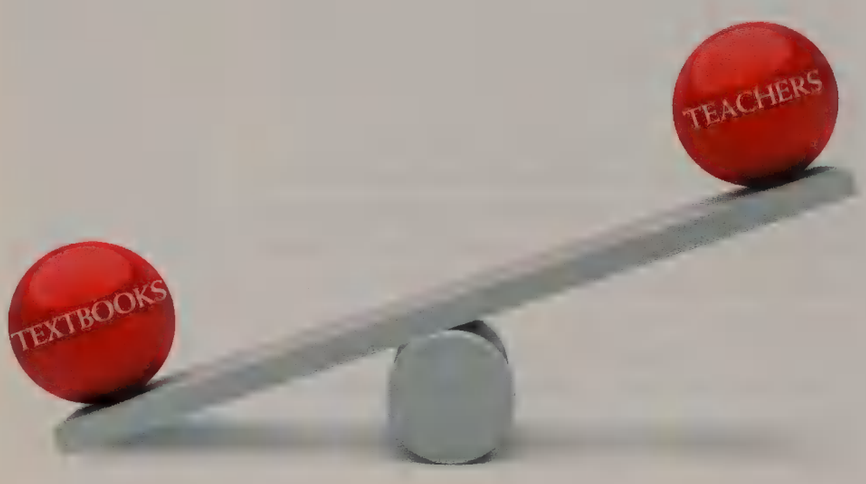
The CCSSM high school modeling standard describes five different actions that students take over the course of a complete modeling task:

1. Identifying essential variables in a situation
2. Formulating models from those variables
3. Performing operations using those models
4. Interpreting the results of those operations
5. Validating the conclusions of those results (CCSSI 2010, p. 72)

The CCSSM describe a sixth modeling action—reporting on the conclusions (p. 73)—that I chose not to consider in this analysis because it seemed within the classroom teacher's locus of control more than the textbook's. For any given modeling task, a teacher may easily request that students report their conclusions in the format of an advisory letter, a written paragraph, or a poster. As we will see, it is much harder for a teacher to fill the absence of the other five modeling actions.

MODELING IN ONE TEXTBOOK SERIES

Each of the five actions represents opportunities to model, and textbooks offer students certain opportunities more than others. I will elaborate on each and report my analysis of how well they are represented in textbooks. (See **fig. 1**.)



Identifying Essential Variables

When students identify essential variables in a situation, for example, they are deciding what information matters to a given task and also what does not matter. Teachers undertake this same process in "adult life" all the time. Let's say that you are debating whether renting or buying a home is more cost effective. What information would matter? Interest rates, rent prices, and home prices would all matter, of course. The temperature outside and the number of letters in the street address are much less consequential, of course. And the consequence of information such as the median income of the neighborhood's residents or the median age of their homes is harder to determine. This process is *modeling*—starting with a context and a question alone and asking, "What information is necessary, and what information is unnecessary?"

In my analysis, these two textbooks most often gave students all the necessary information. In only seven of the eighty-three tasks that I analyzed did textbooks withhold some crucial information or ask students to decide what information would be necessary to answer the question.

Formulating Models

In the example of renting or buying a home, once the essential variables have been identified, you formulate models from those variables. Those models might look like tables illustrating the total cost of ownership of the home and the cumulative amount paid in rent over time. Alternately, the model might look like a graph illustrating where those two functions intersect. They might look like equations that will help you locate that intersection point more precisely than will the graph.

It is important that students both see the need for these models and understand them, which is not to say that students need to discover them without teacher support. It is enough that students should experience how disorganized numbers become without a table to organize them, how opaque those numbers become without a graph to visualize them, and how hard it is to make predictions outside our existing data without an algebraic equation.

In 76 percent of the tasks that I analyzed, either the textbook gave the models in the text of the problem, including equations and tables and graphs in the task before students experienced a need for them, or the tasks referred students back to sample problems that featured worked examples of those same models. In my analysis, only twenty of the eighty-three analyzed textbook tasks withheld the model, asking students to derive the graph, equation, or table themselves.

Without doing that hard work, students might come to regard mathematics as a series of formulas to memorize rather than as a series of models that make sense.

Performing Operations and Interpreting Results

Once you have formulated a model, you then perform operations using those models. In our housing example, you know how many years you intend to live in a home, so you calculate the "total cost of ownership" for both options. A calculator is often essential to this step. Then you take the numbers the calculator produces, which are unitless and divorced from any kind of context (the calculator does not know that it is helping decide on housing), and then you interpret the results of those operations. Those numbers represent money. Their units are dollars. And you should choose the option with the lowest number, not the highest.

It may not surprise readers to find out that textbooks give students ample opportunities to practice these modeling skills. In particular, 86 percent of tasks labeled "modeling" asked students to perform an operation—graphing functions, evaluating numbers in an equation, or filling in rows in a table, for several examples. Eighty-one percent of tasks required students to attach units to their answers or interpret them in some other way.

The comprehensive focus of textbooks on these two skills is at odds with their diminishing importance to people who do mathematical modeling for a living. Companies fight to hire people who can thoughtfully generate models for a changing world (the first two modeling actions), but those models are often programmed into computers that automatically handle the second two modeling actions (performing operations and interpreting their results). Or, as Einstein put it nearly a century ago, "The formulation of the problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill" (Einstein and Infeld 1938, p. 92).

Validating Conclusions

When it comes to validating the conclusions of those results, the mathematician George Box was insightful when he said, "All models are wrong, but some are useful" (Box and Draper 1987, p. 424). The mathematical models that we use to describe the world make assumptions that simply will not bear out with 100 percent accuracy. We assume that a runner is modeled by a constant rate equation, not because that is correct but because it is useful. We assume that a file cabinet is modeled by a rectangular prism when we want to figure out how much paper we can store inside

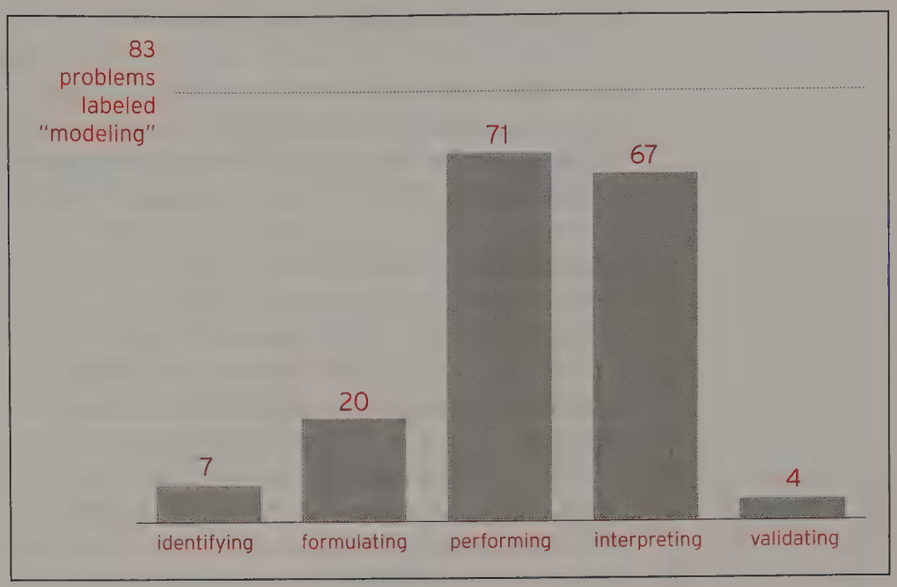


Fig. 1 Modeling opportunities in one textbook series represent different actions.

it, but the prism is smooth and its walls are infinitesimally thin. None of these assumptions is correct, but they are useful.

Sometimes we validate the conclusions of these decisions by performing an internal error check, asking ourselves, "Does this make sense? Have I forgotten anything?" and correcting our models. But sometimes we validate our models when we watch how well they predict—when we find out the runner's actual time for the race or when we try to stuff all this paper we bought into the cabinet. When meteorologists predict the weather, they get to watch the next day to see if their models for weather were valid. The same goes for stock analysts, flight controllers, home buyers—and the list goes on.

The crucial point is that all these models will be wrong. Only in mathematical textbooks do we insist that mathematical models have a zero percent margin of error. We make that warranty when we construct answer keys that match our students' calculated answers without exception. We do students a disservice when we tell them that anything short of 100 percent accuracy means that they have made a mistake. Modeling with zero percent error does not occur in the world of professional modeling or the world of modeling as described in the CCSSM.

Through my analysis, I identified only four tasks out of eighty-three in which the textbook invited students to consider the difference between the conclusions of their mathematical models and real-world results. All four of those tasks were presented in the probability cluster. On the context of probability, these textbooks admitted to students that our mathematical models are not a perfect reflection of the world and that students should verify them experimentally.

RECOMMENDATIONS FOR TEACHERS

The promise of mathematical modeling as written about in the CCSSM and as experienced by professionals has not been realized in these two textbooks, in spite of their claims. These textbooks are strongest where textbooks have traditionally been strongest—in asking students to perform operations and then to attach some cursory meaning to the results of those operations.

So what can we do? The graph in **figure 1** shows the opportunity for teachers. We should step into those gaps and provide our students with more, and more diverse, opportunities to model. Textbooks well emphasize the modeling actions of "performing operations" and "interpreting results"; we teachers need to emphasize the other three modeling actions. I will offer here some suggestions for identifying essential variables in a situation, formulating models from those variables, and validating the conclusions of those results.

If you want to provide your students opportunities to identify essential variables, ask them the question, "What information is necessary here?" whenever you can. It is a question that adults ask themselves intuitively in their grown-up lives all the time. Whenever we encounter a nontrivial challenge, we take our question and then start informing it with data—for instance, "Which grocery line should I get into?" What information matters here? What information matters most? Ask your students, "Should I buy a year-long parking pass or just pay for each day?" Again, the question cannot be answered without more information.

One task in the analyzed textbooks that adequately addressed this modeling action asked students to come up with a way to compare athletes from the same sport. For instance, compare two point guards in basketball. Ask your students, "What information matters here?" Points scored? Definitely. Turnovers and rebounds? Probably. Shoe size and hair color? Likely not, but deciding on essential information requires the joint action of deciding what information is inessential. Would students think that information such as player age, past injuries, and salary is essential or not?

Feel free to stop the conversation there. Do not feel obliged to move through the other modeling actions. It is enough that you have asked and that your students have answered the very grown-up question, "What information do I need here?"

When it comes to formulating models from those variables, we must not think that students need to discover all these complicated mathematical models, although some of them are discoverable. With our point guards, students can ask themselves, "Now that I have identified points, rebounds, and turnovers as my essential variables, what do I do with them to answer my question?" Perhaps the model is as simple as this equation:

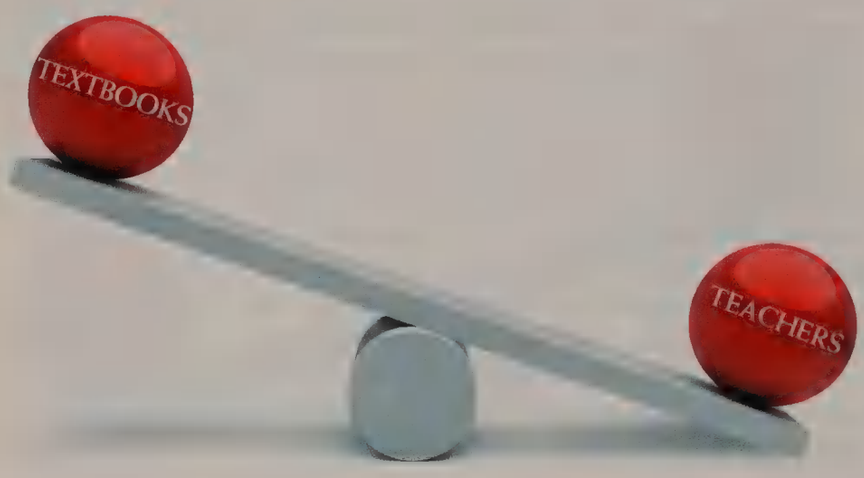
$$\text{player quality} = \text{points} + \text{rebounds} + \text{turnovers}$$

Students may see that this will be a poor model for point guard quality because although points and rebounds are an indicator of player quality, turnovers are bad, and yet they are all added to the model. Rebounds are also good but perhaps not as good as points, so we might revise our model like this:

$$\text{player quality} = \text{points} + 0.5 \cdot \text{rebounds} - \text{turnovers}$$

Other models are formulated less intuitively, and students will need our help constructing them. My precondition for talking about new models is that students should see the "need" for them (Harel 2013), which often occurs when students watch their old models fail them. Proportional models were easy for my students, but a race with a head start required a linear model. Then linear models became easy, but the path of a basketball through the air was modeled very poorly by a linear graph. Once students see their old models fail, we have the opportunity to talk about new models. If students do not see their old models fail, then mathematics becomes a very long tour through all these models that their teachers cherish for no reason that the students can easily discern.

After students have performed operations on a model and interpreted the results of those operations, they will have a prediction. For example, their model may predict that a water tank will empty in eight minutes; their model may predict that the fuel in a car will last only until the next gas station; their model may predict that they will need to buy fourteen pizzas to feed everybody at the party. At this point, students may reasonably



wonder, "Is my prediction correct? How well has mathematical modeling served me here?" Here we should validate students' conclusions by showing the outcome of the event. Show a video of how long it actually took the water tank to fill up. Show the gas gauge at the next gas station. Show a picture of the empty pizza boxes at the end of the party. The world will rarely cooperate completely with our mathematical models; there will be a few leftover slices. But seeing some documentation of their difference—with a photograph, video, personal testimony, or a live demonstration—will help students understand the true of power of mathematical modeling, one of the most powerful tools humans have for understanding their world, although not omnipotent.

Let's worry less about performing operations and interpreting results. Not because they are not important but because the gravity of our textbooks already pulls in their direction. Instead, let's help students see that the world will rarely fully validate the conclusions drawn from their models, that some uncertainty is to be expected, that mathematics is smooth and frictionless, whereas the world is rough and full of surprises. We do that by letting them see the result of their modeling rather than simply telling them those results.

FILLING IN THE GAPS

Not every mathematics task can or should engage students in all five modeling actions. It is possible to practice these actions in isolation from one another also. But this analysis suggests that textbooks offer students limited opportunities to model, tending to complete the first two actions for students (identifying variables and formulating models) while ignoring the last action (validating conclusions). Teachers must make up the difference, offering opportunities for students to engage in those five actions and also to implement tasks that occasionally require all five in concert. (See, for example, Three Act Tasks [threeacts.mrmeyer.com] and 101 Questions [101qs.com]).

I began this article by asking, "What does the CCSSM call 'modeling'?" and "Do textbooks do that definition justice or not?" The answer to the first question is rather clear. The CCSSM authors, working in territory mapped by Box and other applied mathematicians, have provided a precise definition of a complicated and muscular mental act. The answer to the second question is far less certain than I had hoped. My conclusion is not that textbook publishers are diligent or lazy, smart or dumb. My conclusion is that the two textbooks I examined, and arguably numerous others, offer certain modeling opportunities for students and teachers, whereas other modeling opportunities simply are not available for students to learn and teachers to teach.

We must work now to offer students those missed opportunities to model—the kind that lured me into mathematics and kept me in mathematics education, the kind valued by their science classes and valued by employers in satisfying jobs. We can work by asking these different questions in our mathematics classes and making ourselves comfortable with answers from students that will be unpredictable and interesting. We can also work by asking textbook publishers to focus their enormous resources in the direction of interesting and complete mathematical modeling tasks. The CCSSM Publishers Criteria are not binding on any publisher, but our textbook procurement budgets are. Our voice has always been the strongest lever we have for extracting the textbooks we need from publishers. It is time we use it.

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Weaving Geometry and

Michelle Cetner

When thinking about student reasoning and sense making, we teachers must consider the nature of tasks given to students along with how we plan to use the tasks in the classroom. Students should be presented with tasks in a way that encourages them to draw connections between algebraic and geometric concepts. To this end, a dynamic geometry program can be used with high school mathematics students to convey standard algebra and geometry concepts in a connected way through the use of carefully constructed questions and directed discussions. The platform used in the following examples is The Geometer's Sketchpad®, but the tasks could be adapted to any dynamic geometry software platform.

The use of tasks that encourage both visual and analytic reasoning has been shown to enable students to be better problem solvers and lead to better long-term conceptual memory (Banchoff 2008; Chinnappan and Lawson 2005; NCTM 2003; Noss, Healy, and Hoyles 1997). One reason may be that the concurrent and connected use of both visual and analytic thinking about a problem encourages more connections to be made between the left and right halves of the brain (Battista 2007; Thornton

Algebra Together

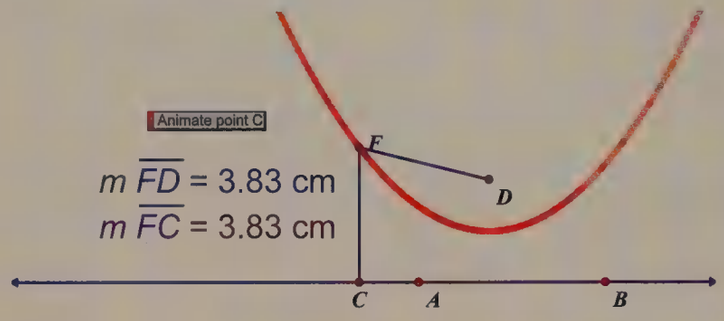
Two activities help create a tapestry of connections.

2001). Engaging students with connected activities in which both visual and analytic reasoning skills are used instead of giving students tasks that are entirely of one nature or the other greatly benefits all students, particularly English language learners and those with multiple learning styles (Thornton 2001).

In *Principles and Standards for School Mathematics* (NCTM 2000), the Connections Standard includes recognizing and understanding connections among mathematical ideas and how those ideas build on one another to produce a coherent whole (pp. 64–66). The Common Core State Standards for Mathematics (CCSSI 2010) do not make any broad statements about connecting different mathematical concepts, but making meaningful connections between mathematical strands is implied through its discussion of the mathematical practice standards of modeling, structure, and repeated reasoning (e.g., pp. 8–72), along with numerous specific examples within the content standards portion of the document (e.g., pp. 66–67, 69). Together, these documents demonstrate the mathematics education community's commitment to helping students form deep mathematical connections across all topics in school mathematics.

Two ideas prevail about the teaching of geometry

THE LOCUS OF POINTS EQUIDISTANT BETWEEN A POINT AND A LINE



1. Select the arrow tool and then **Edit**, **Preferences**, and **Text**. Check the box labeled as **For all new points** and **OK**.
2. Select the point tool and plot two points near the bottom of your sketch.
3. Select the arrow tool, both points, and **Construct**, **Line**; then **Construct**, **Point On Line**.
4. Select the point tool and plot a point above line *AB*.
5. Select the arrow tool, points *D* and *C*, **Construct**, **Segment**; then **Construct**, **Midpoint**, segment *DC*, **Construct**, **Perpendicular Line**, and deselect.
6. Select point *C*, line *AB*, **Construct**, **Perpendicular Line**, the perpendicular line through *E*, **Construct**, **Intersection**, point *D*, **Construct**, **Segment**; **Measure**, **Length**, and deselect.
7. Select points *F* and *C*, **Construct**, **Segment**; **Measure**, **Length**.
8. Select lines *EF* and *CF*, segment *DC*, point *E*, **Display** and **Hide Objects**.
9. Select point *F*, **Display**, **Trace Intersection**, and deselect.
10. Select point *C*, **Edit**, **Action Buttons**, and **Animation**. Then select the tab named **Label**; type **Animate point C** and **OK**.
11. Click on the **Animate Point C** button to observe the shape that is being generated.
12. Select the **Text** tool; double-click to open a dialog box. Then answer the following questions:
    (*a*) Describe the relationship between the segments *FC* and *FD*.
    (*b*) Describe the relationship between point *F* and the objects point *D* and line *AB*.
    (*c*) If the curved shape that you made is called a parabola, write a geometric definition for the term *parabola*.

Source: Adapted from Cinco and Eyshinsky (2006)

Fig. 1 Steps to create a parabola on The Geometer's Sketchpad are listed.

and algebra in a connected way. The first is that geometry acts as a motivator in teaching algebra because visualization and demonstration of patterns can be instrumental in teaching students how to generalize and think algebraically (Presmeg 2006; Saha, Ayub, and Tarmizi 2010). Although this perspective is useful in many situations in an algebra class, this article will focus on the second idea—that it is possible to teach both geometry and algebra concurrently and in an integrated way (NCTM 2003). Although the definition of *integrated* varies widely among teachers and schools, a survey of teachers (NCTM 2003) revealed that teachers often emphasize connections between subjects and topics within a subject through problem solving and by using mathematical reasoning and technology.

With the availability and versatility of today's technology, we can help students make powerful connections between algebraic and geometric concepts. Through the use of dynamic software, we can construct classroom activities that will help students draw such connections, encouraging the shift to higher levels of student reasoning. For example, in the exterior angle activity discussed (task 2), the use of the click-and-drag feature of the program makes it possible to manipulate an otherwise static image, creating a dynamic number experience for students. Students actually see the number change as the figure changes and thus have an enduring image that they would otherwise not have (Mackrell 2011; Noss, Healy, and Hoyles 1997). Seeing the figure and number change concurrently can help all students draw deeper connections with higher levels of reasoning about variable, incorporating the visual into every stage of the activity (Saha, Ayub, and Tarmizi 2010).

The availability of technology, however, will have little bearing on student learning without the influence and encouragement of the teacher (Lu 2008). Teachers should help students move back and forth, relating the analytic with the perceptual investigations and the rigorous with the intuitive approaches (Noss, Healy, and Hoyles 1997); in addition, they should help students differentiate between *convincing evidence* and *proof* and understand when each is appropriate (Olive 2000). The tasks in and of themselves are relatively standard lessons in ninth- or tenth-grade mathematics curricula, but the teacher's implementation and guidance through careful questioning enhances the number and nature of connections that students will make.

TASK 1: THE CONSTRUCTION AND EQUATION OF A PARABOLA

This task uses a geometric lab approach to show that the set of points equidistant to a point and a line is a parabola. This step-by-step type of activity (see **fig. 1**), adapted from Cinco and Eyshinsky (2006), guides those who are just learning to use The Geometer's Sketchpad (GSP). This task addresses the following Common Core standards: "Derive the equation of a parabola given a focus and directrix" (CCSSI 2010, p. 78) and "Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function" (CCSSI 2010, p. 69).

This activity asks students to construct and recognize a parabola (steps 1–11) and to create a definition for the parabola that they constructed. Students will generally recognize the shape of the parabola quickly but will need more guided discussion to construct the geometric definition (in step 12), even after answering the first question on the worksheet: Describe the relationship between segments $\overline{FC}$ and $\overline{FD}$. The teacher may encourage them by rewording the second question: "If $\overline{FC}$ is congruent to $\overline{FD}$, what does that tell you about the distance from F to D and from F to $\overleftrightarrow{AB}$?" From this, students should determine that the distances are equal.

Before moving on to the third question, the teacher may ask, "Is this always true? What if you move point F?" The exchange that follows may begin with the student conjecturing that the relationship holds because you can see the measurements remain the same when you click and drag. The discussion may evolve into students trying to deductively convince either the teacher or one another about features of the diagram, such as that $\overline{FC}$ and $\overline{FD}$ have to always have the same length because E is the midpoint of $\overline{CD}$. Depending on the students' understanding as well as time constraints and the general purpose of the lesson, the teacher could decide to develop this discussion further or include other observations about parts of the parabola, such as the vertex and the axis of symmetry, and why they will always be present.

Students are generally eager to write the answers to all the questions posed on their worksheet. As a result, they will invariably bring the conversation back to the third question—how to write the geometric definition for the trace that they created. By now students understand that a parabola is the set of all points equidistant from a given point and line and are ready to move on.

Making Connections

One valuable aspect of dynamic geometry software is that students can then take this geometric figure that they have constructed and overlay coordinate axes. The teacher may encourage students to describe the parabola analytically by asking them to conjecture about what the equation of this function should be. Students may ask where they should put the axes, leading to a discussion about the rigidity of the parabola $y = a(x - h)^2 + k$, in which the shape (wideness or narrowness) of the parabola completely depends on a. Placing the origin at the vertex of the parabola makes a much neater equation: $y = ax^2$. This realization would ideally be developed by having students experiment with and discuss the placement of the origin on their own constructed sketches.

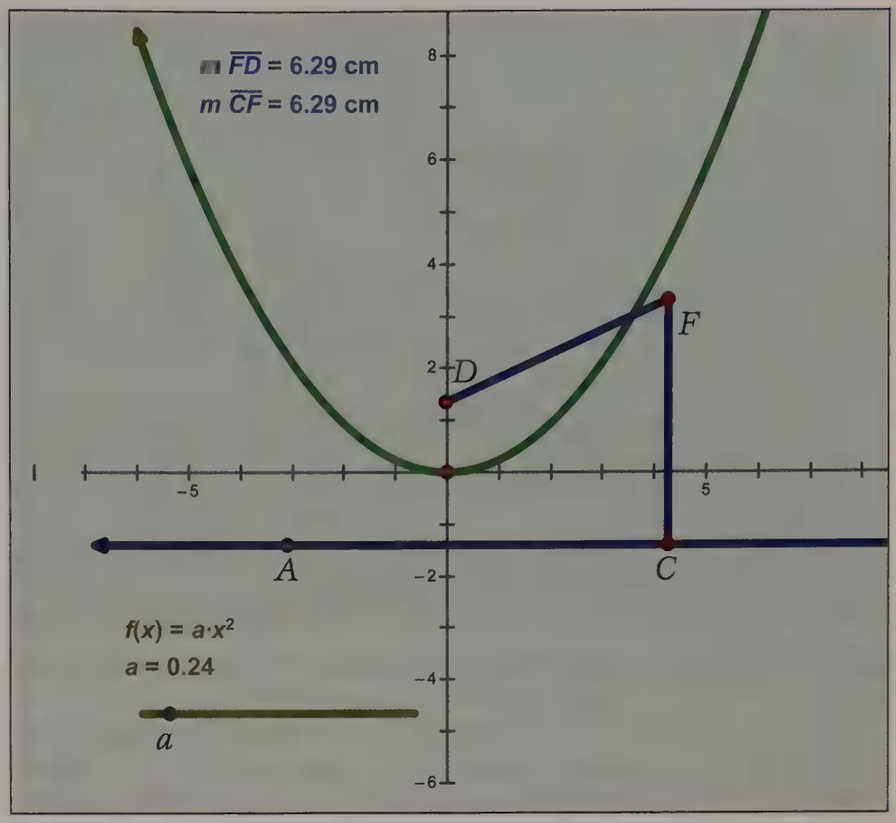


Fig. 2 A slider for a is used to graph $f(x) = ax^2$. Adjusting the value of a opens the parabola to include point F.

Depending on their skill and comfort level with the software, students may plot the function $y = ax^2$ for different values of a to guess and check, or else they may construct a parameter and slider for a so that they can observe the changes in the graph more easily. Once students have the correct equation, their function plot overlays the trace exactly (see **fig. 2**). This "geometric approach enables students to manipulate functions as an entity," making new connections that would not be possible by using an analytic or numeric approach alone (Mousoulides and Gagatsis 2004, p. 391).

Extending the Task

Depending on the focus of the class, the lesson may stop here or go on to explore an analytic proof of the parabola's equation using the Pythagorean theorem or distance formula. Because our parabola has its vertex at (0, 0) and students already defined it as the set of points equidistant from the focus and directrix, they can define the coordinates of the focus D $(0, p)$ and the equation of the directrix $y = -p$. They

The discussion may evolve into students trying to deductively convince either the teacher or one another about features of the diagram.

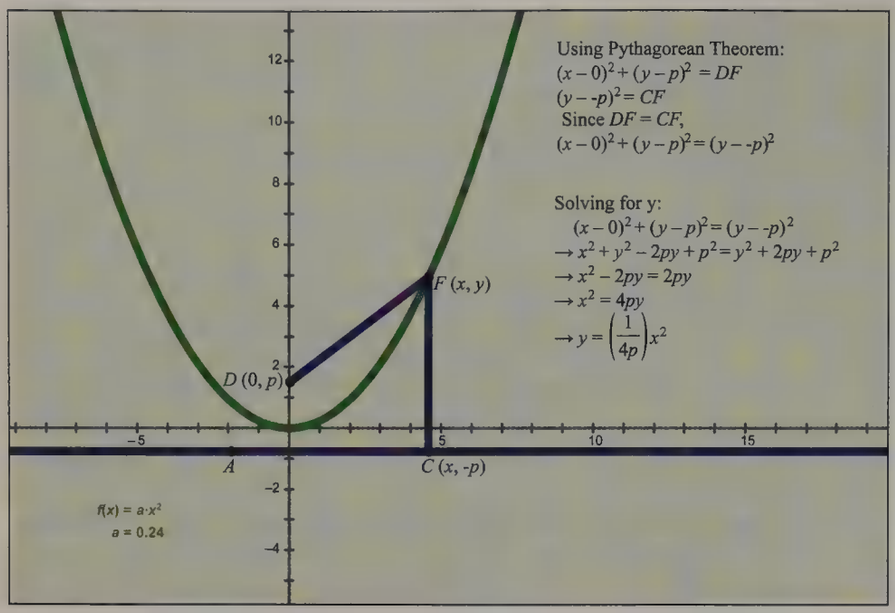


Fig. 3 The geometric features of the parabola—its focus and directrix—are related to its equation.

could then explore further to use a combination of analytic proof and geometric conjecture about what effect the coordinates of the focus and the point C on the directrix have on the equation (see **fig. 3**).

The equation that students develop, $y = 1/(4p)x^2$, shows that the a in $y = ax^2$ is given by the expression $1/(4p)$. So the width of the parabola, controlled by a, is inversely related to the distance, $2p$, between the focus and directrix. Advanced students may be encouraged to explore the same methods when the vertex is not at the origin or even when the directrix is not horizontal.

This task could be extended to calculus by thinking of the relationship of the tangent line that the student constructed to the curve (hidden in the figure). Students might measure the slope of the line and compare that with what they already know about the derivative of the function at a point. They also might go on to argue mathematically why this line ends up being the tangent line.

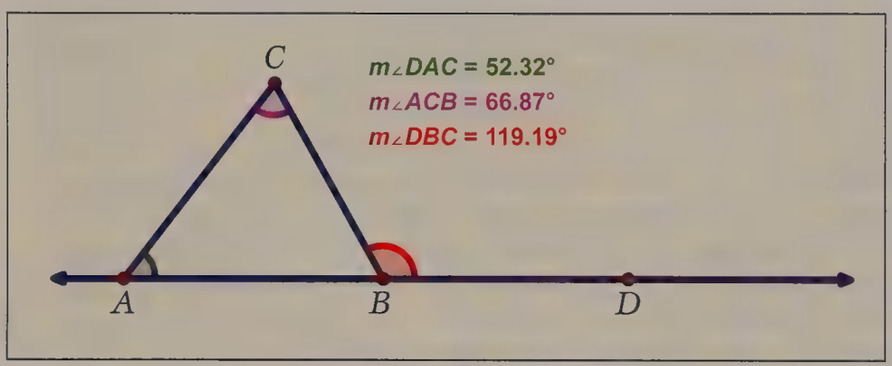


Fig. 4 Angle measures exemplify the exterior angle theorem.

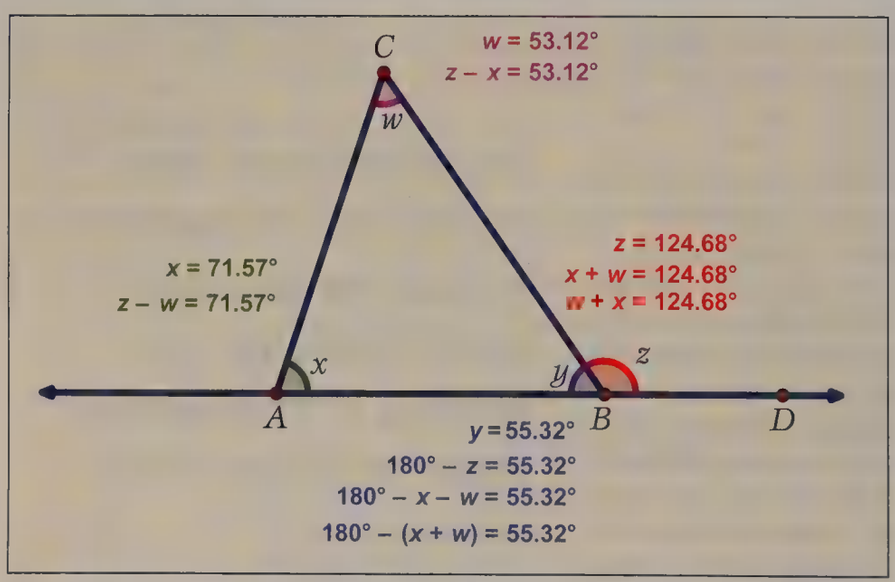


Fig. 5 Equivalent expressions, grouped by color, are evaluated as the same measurement.

TASK 2: EXTERIOR ANGLE OF A TRIANGLE

This activity is designed as a teacher-led classroom demonstration but could also be given to students as a preconstructed sketch, written as a lab, or generated by students as in the previous activity. The purpose of this activity is to teach that the measure of the exterior angle of a triangle equals the sum of the measures of the alternate interior angles (see **fig. 4**). However, equally important in this lesson is the algebraic interpretation and representation of the situation with expressions and equations. This activity addresses the following Common Core standards: "Prove theorems about triangles" (CCSSI 2010, p. 77) and "Interpret expressions that represent a quantity in terms of its context" (CCSSI 2010, p. 64).

The teacher may initiate this lesson by asking students to conjecture and discuss what relationships exist within the diagram. One technique for fielding student responses is to have GSP calculate different combinations or measurements as students suggest and then ask students to classify and sort the calculations that they generated.

Making Connections

Many students do not have a good understanding of variable, and activities such as this one uses the concept of dynamic number to reinforce the concept that variables change (Mackrell 2011; Olive 2000). For example, the teacher may encourage students to click and drag any point in the diagram and discuss what happens to the numbers. Students will notice that the measure of the exterior angle DBC is equal to the sum of the measures of the two interior angles. Beyond that, they should notice that the angle measures change along with the diagram but that they keep the same relationship. This connection between the geometric and the numeric has a profound effect on student understanding of dynamic number.

To develop student understanding of the equivalent expressions that are present, it also helps to

change the labels for the angles in front of students so that they can tell that naming the angles by one letter, three letters, or even words does not change their relationships. Through a class discussion and dynamic manipulation, students may also see equivalent expressions, such as $180 - z$ and $180 - (x + w)$ (see **fig. 5**).

Finally, we want students to think creatively about arguments that would constitute a formal proof of their conjectures. As in the parabola activity above, students should compare, contrast, argue, and evaluate their reasoning. For example, a student might intuitively know that the angle measure labeled z is the same as the sum of angle measures x and w; however, in finding ways to convince others that this conjecture is true, the student will learn to think logically through an argument.

One student may cite the theorem that the sum of the measures of the angles in a triangle is 180°, as is the sum of the measures of the angles in a linear pair. Geometrically, the angle ABC, common to both these relationships, leads to a proof of the conjecture about the exterior angle. Another student may use the blue set of equations in **figure 5** to show the same theorem algebraically.

Extending the Task

The teacher may take the task one step further and ask other students which argument they prefer, whether and how the arguments are the same or different, and how they could use one argument to support the other. This discussion tends to lead students away from the get-the-answer-as-quickly-as-possible approach to problem solving and in the direction of reasoning and sense making about generalizations in mathematics.

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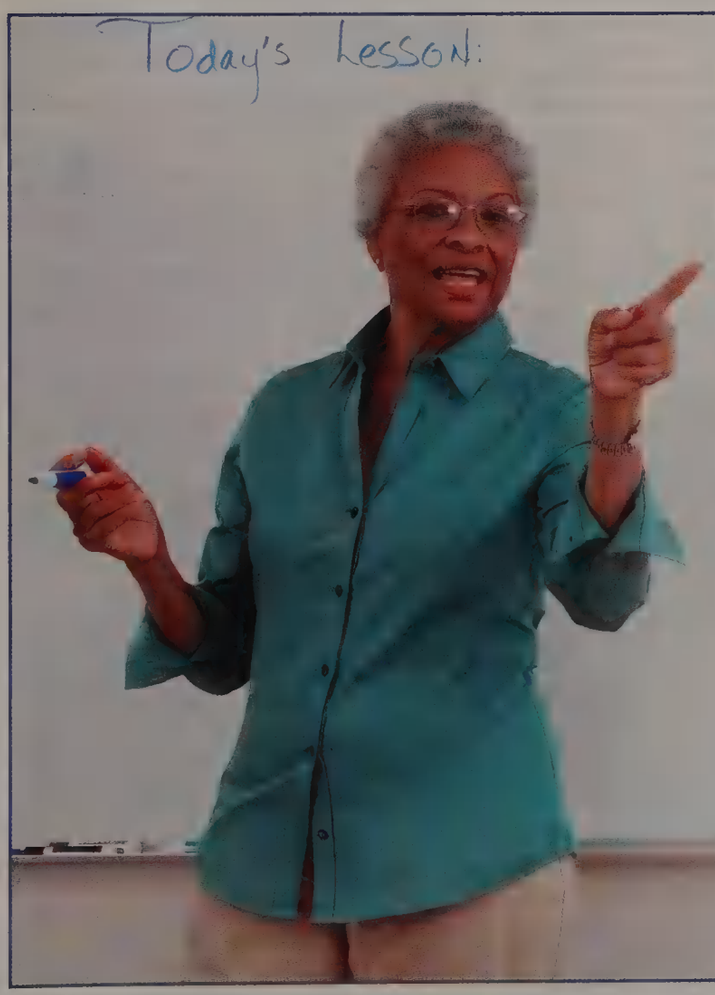
THE TEACHER'S ROLE

These two tasks demonstrate two of many ways in which teachers can use dynamic geometry software to help students learn to draw deep connections between geometry and algebra. The teacher's role as classroom facilitator is crucial in encouraging students to reason, generalize, question, and prove their conjectures.



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One responsibility that we have as teachers is to constantly ask our students important questions—Why?" "How?" "Where?" "When?" and "How do you know?"—and encourage them to ask themselves the same questions as they mature mathematically. Students need to know that mathematics is more than just a dry list of formulas and procedures to memorize and imitate. Making creative mathematical connections enhances the ability to model and understand the world around us.

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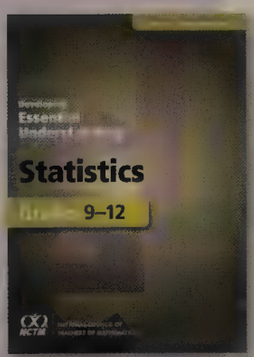

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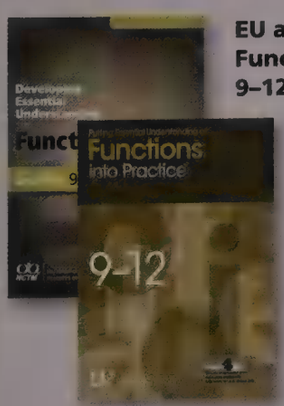

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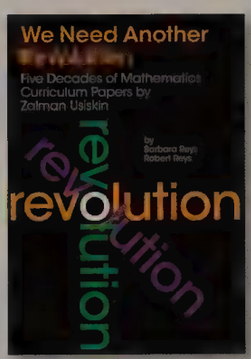

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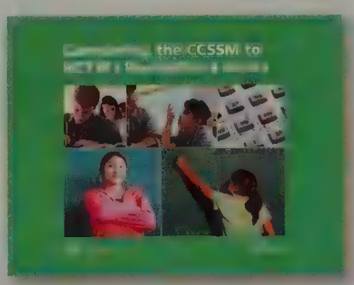

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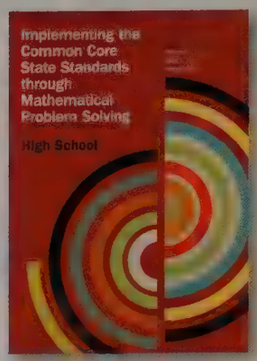

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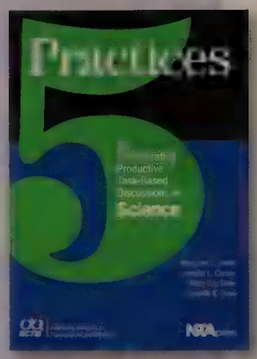

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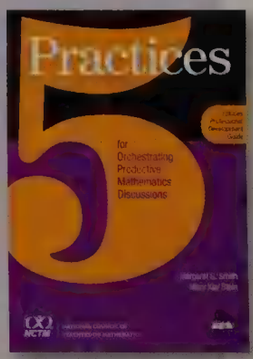

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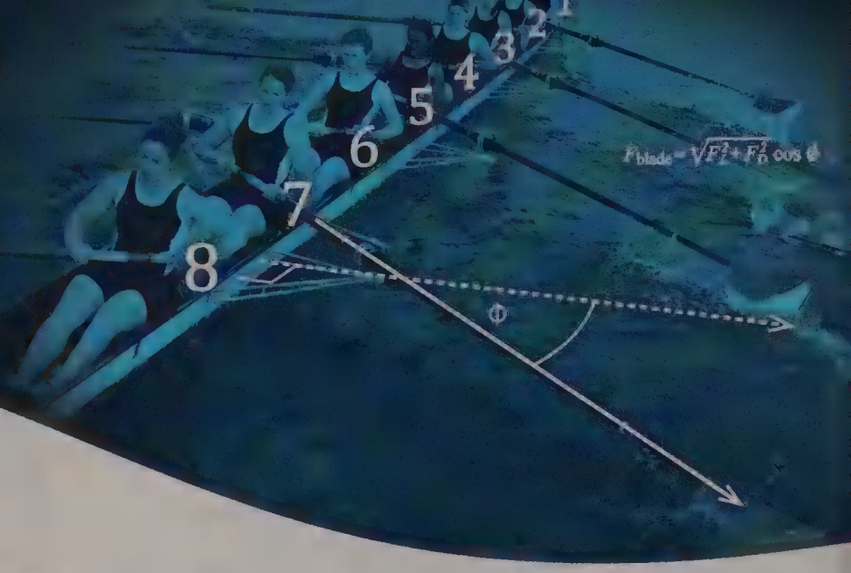

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Presented by Beth Skipper, TCM editor

Thursday, April 16:
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Saturday, April 18:
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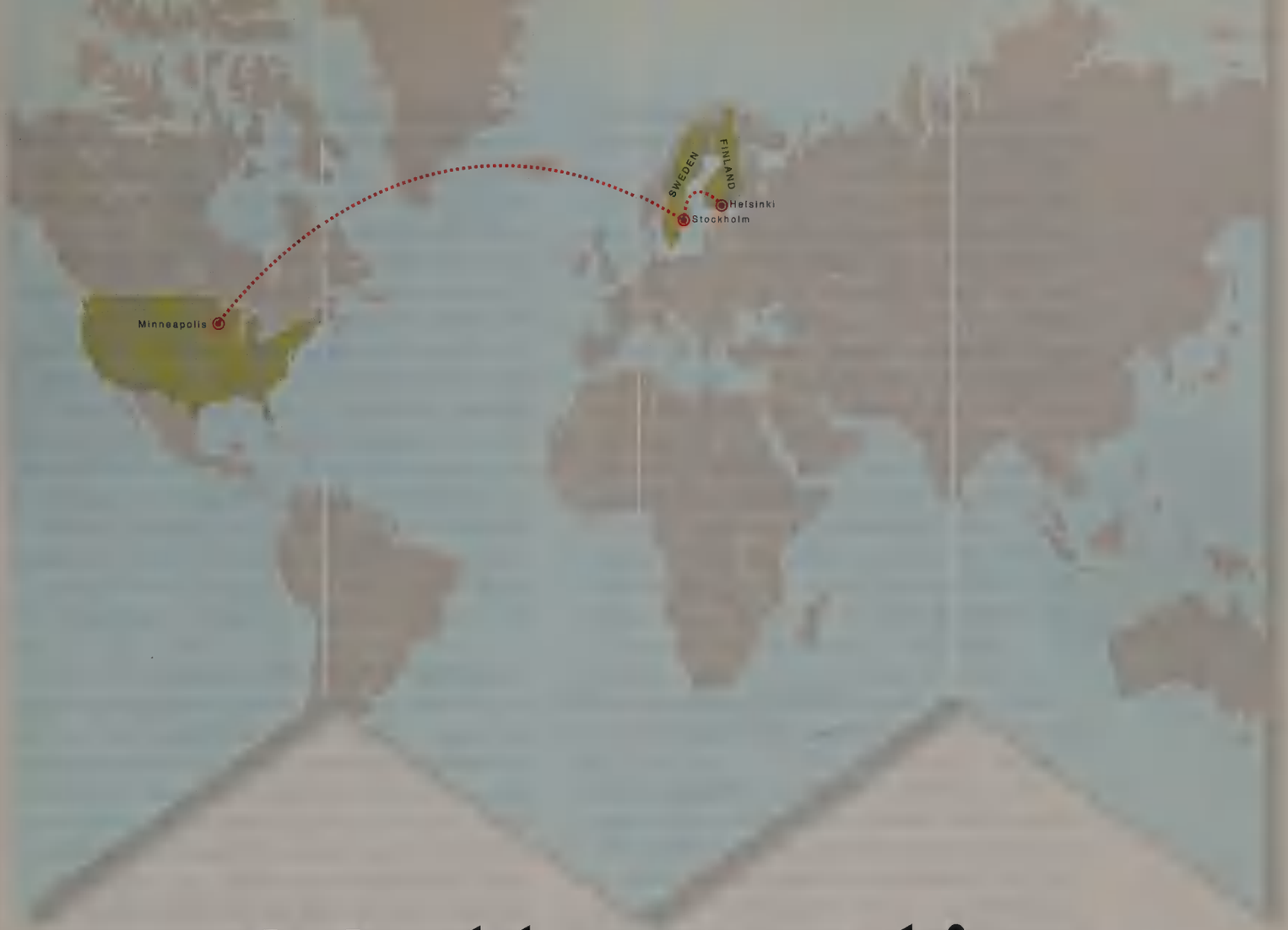
MATHEMATICS teacher | mathematics teaching in the MIDDLE SCHOOL | teaching children mathematics



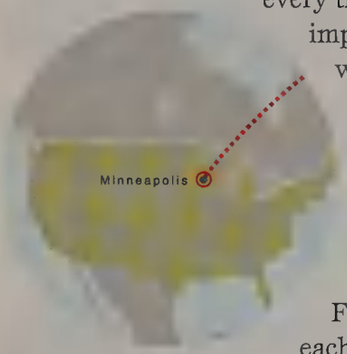
Mathematics Lessons from Finland and Sweden

International comparisons highlight many differences and a few similarities among these two countries and the United States.

Rebecca L. Seaberg

When responding to a question about why mathematics education is going so well in Finland, a Finn might just answer, "Tämä on Suomi!" ("This is Finland!"). When a Finnish upper-secondary mathematics teacher asked his quiet students, who were busy taking notes, "Hur många element i den aritmatiska talföljden {6, 21, 36, 51, . . .} är mindre än 10000?" at least half the students raised their hands with ideas. The question translates as "How many elements in this arithmetic sequence {6, 21, 36, 51, . . .} are less than 10,000?" (Some Finnish schools are Swedish speaking.)

This was one of many mathematics lessons I observed in Finland and Sweden during spring 2013 while on a sabbatical from teaching. Finland has become known worldwide for its steady improvements in its educational system over the past forty years and for high scores in reading, mathematics, and science according to the Organization for Economic Co-operation and Development's (OECD) Program for International Student Assessment (PISA) of fifteen-year-olds conducted every three years since 2000. What makes this improvement noteworthy is that it has occurred without the extra hours of after-school and weekend classes that students attend in other high-performing countries such as Korea, Japan, Singapore, and China (Schachter 2011, p. 42). During my sabbatical, I visited six different schools (all grades 7 to 12) in the Helsinki area in Finland and sat in on two to three classes in each school. I was also able to visit with a mathematics education professor at the University of Helsinki.



I then crossed the Baltic Sea and made similar visits to three Swedish schools. Sweden's educational system is structured similarly to Finland's, yet its PISA results in mathematics have been closer to those of the United States, which have consistently been below the OECD average. One of the principals at Lindeskolan, a gymnasium (an upper-secondary school) in Lindesberg, Sweden, arranged a variety of experiences for me, including visiting advanced mathematics classes, a mathematics class for nursing students, and mathematics courses for immigrant students who need remedial work to prepare them for regular gymnasium programs. I also visited elementary schools in Lindesberg and Attarpsskolan, the latter a school for grades 1–9 near Jönköping.

THE FINNISH AND SWEDISH SCHOOL SYSTEMS

Finland and Sweden both require nine years of comprehensive school. Children enter first grade at the age of seven (most children also attend a year of preschool at age six). Grades 1–6 are called elementary grades, and grades 7–9 are called lower-secondary grades. These two levels can be housed in separate schools or together. The lower-secondary grades are similar to the middle school grades in the United States and are taught by teachers who majored in the subject or subjects that they teach. Almost all students go on to more schooling after ninth grade because of the significant advantage for future employment.

Grades 10–12 in Finland consist of either general or vocational upper-secondary school. General upper-secondary school is focused on academics, and most students go on to attend a university after finishing. To move on to a university in Finland, students must take the Matriculation Exam (ME) at the end of upper-secondary school.

Approximately 40 percent of Finnish students choose to attend vocational upper-secondary school. If a student chooses vocational school, he or she would typically continue on to a polytechnic school (which could be defined as a university of applied science). In Sweden, students can choose from upper-secondary schools all over the country, and some specialize more in certain fields of study. Vocational choices are also available at Swedish upper-secondary schools. In both countries, this level of school is called a gymnasium, and entrance at a particular school of choice is competitive, with acceptance based on grades from the first nine years.

CLASSROOM OBSERVATIONS

So what did I observe? In many ways, mathematics classrooms in Finland and Sweden are very similar to what would be considered traditional classrooms in the United States. Classes began with checking homework and questions, followed by the teacher giving instruction in the new material, and ended with students working on their new assignment. But there were interesting differences as well, in mathematics and the education systems in general.

Finnish mathematics classrooms place a strong emphasis on problem solving. According to Perkkilä and Lehtelä (2007, p. 83), "The Core Curriculum for the Finnish comprehensive school lays emphasis on some key concepts like everyday applications, problem solving, and the student's own thinking, experimental learning and the student's active role in acquiring information."

Resources

Consistent with this emphasis, Finnish upper-secondary students are provided with a book of formulas and tables for mathematics, physics, and chemistry. Instead of testing for retrieval of facts, the focus is higher-level thinking. The students are allowed to use the reference book for all work, including the Matriculation Exam that upper-secondary school students take before graduation. Because of this common resource, the textbooks did seem to have a greater focus on problem solving and applications.

Mathematics textbooks in Finland have *all* the answers in the back of the book, so students can always check to see whether they have gotten the right answer. Sweden is similar, except that the books also contain some solutions in the back instead of just answers.

The National Curriculum in Finland provides guidance and necessary regulations for all schools,

but curriculum planning is done on the local level. It is the responsibility of teachers, principals, and municipalities to implement the core curriculum. Schools may choose from a variety of Finnish language mathematics texts. (One drawback for Swedish-speaking schools in Finland is that at this time there is only one choice in Swedish.)

I observed some unique differences in Swedish schools. Sweden has more immigrants than Finland (although immigration is growing in Finland) and works hard to help prepare them for success in Swedish schools. I observed and visited with a mathematics teacher who works exclusively with immigrants and who has developed many of her own materials to help her diverse group of students.

Technology

Interactive whiteboards were in every Finnish classroom I visited, although I observed only one teacher taking advantage of this tool's capability (in an upper-secondary course in vectors). Everywhere else the board was used as a place to display the textbook or notes with a document camera. Finns acknowledge that there is some resistance to using technology in the classroom. Leino and Furinghetti (2007, p. 187) state, "Though the results of computer-based teaching are good in several experimental projects, relatively few changes can be found in Finnish mathematics teaching. Teachers say that they do not have enough time to learn and teach the use of the programs developed."

I found interactive whiteboards in only a few Swedish classrooms and was told that a decision was made not to invest in them at the upper-secondary school in Lindeskolan.

Texas Instruments calculators were used in classrooms, in particular the TI-Nspire CX CAS in the Finnish integral calculus course I visited. I observed students using laptop computers in only one class, an eighth-grade mathematics class. Students in groups of three or four used plastic clinometers to measure the height of a tree of their choice on the school grounds (Finland has an ample supply of tall pines!) and created a report with photographs and diagrams using school cameras and laptops. This teacher also teaches information and communications technology (ICT).

Less Is More

Another phrase the Finns use is *vähemmän on enemmän* ("less is more"). Finnish students spend about 5500 hours in school between the ages of seven and fourteen, whereas estimates from some states in the United States are closer to 7500. In lower-secondary schools, on average, Finnish teachers teach about 600 hours annually, or about four teaching lessons daily. According to the OECD, the average annual total teaching time in the United States in the lower-secondary grades is 1080 hours, which involves six or more daily lessons. So Finnish teachers have more time in their day to plan curriculum, assess learning, provide remedial help, and reflect on their teaching (Sahlberg 2011, pp. 63–64).

I found schools in both Finland and Sweden to have a more relaxed atmosphere than in the United States, with at least fifteen minutes between classes. I was always invited to join other teachers for a cup of coffee during this time. Most secondary classes met for seventy-five minutes, two or three times a week. I observed relatively small class sizes in both Finland and Sweden. The average class size in Finland was fifteen, and the largest class I observed was a Finnish advanced ninth-grade class with twenty-eight students.

NATIONAL PRIORITIES

One of Finland's highest goals is equity in their schools and between schools. Finns call it *nosto monet* ("lifting the many"). They emphasize cooperation between schools and communities instead of competition. One result of this philosophy is very little tracking in mathematics classes. The only tracking that I observed was in the ninth grade, which is the final year of comprehensive school. Tracking can vary by school, however, so it is up to each school or community to decide whether to offer different levels during ninth grade.

If a school does offer tracking, students and families are free to decide what level of mathematics course they would like to take. The teachers or school staff can give advice, but they do not choose for the student. The reason that I was given for tracking in ninth grade is to help prepare students for the different paths they may take following comprehensive school. One criticism that I heard from Finnish mathematics teachers is that they feel that they do not always challenge the top students as well as they could because of their focus on the "many." One said that their top 10 percent of students may achieve less but their lower 10 percent of students achieve more. Results from the

The educational system in Sweden is more accommodating to students' needs, whereas in Finland the priority seems to be that everyone should learn the subjects well.

Finland's high requirements for entering teacher education programs in universities mean that beginning teachers must earn a master's degree.

2013 International Mathematics Olympiad support this comment: Finland ranked sixty-seventh out of ninety-seven countries participating. The United States placed third, after China (first) and South Korea (second); Sweden was sixtieth (Dolinar 2013).

In Sweden, I was impressed with an emphasis on oral communication within the mathematics curriculum that I did not find in Finland. I sat in on a class in which the students were presenting their solution to a problem to a small group of classmates and their teacher, a required part of the evaluation of the students by the national curriculum. The upper-secondary schools in Sweden resemble a community college in many ways. The students have more breaks in their day, are free to come and go, and are already focused on a specific field of study. There are a wide variety of majors, such as nursing, electricity and energy, restaurants and food, natural science, humanities, and child care. Many courses at the beginning of each program are common courses (for example, English, history, religion, mathematics) so that students can switch from one major to another without too much difficulty.

Both a Finnish teacher who had visited schools in Sweden and Swedish teachers described the focus in Sweden as more accommodating to the students' needs, whereas in Finland the priority seemed to be that everyone should learn the subjects well.

STUDENT PREPARATION

Hendrickson (2012, p. 488) highlighted the predominant use of formative assessment in Finland that aids student learning, in contrast to high-stakes standardized assessments in the United States that are used mainly to rank students, teachers, and schools.

The only standardized assessment in Finland, the Matriculation Examination (ME), is taken at the end of grade 12 in upper-secondary school and is a significant factor in acceptance to university programs. In contrast with the large number of multiple-choice type of questions on U.S. college entrance exams, the ME is essay based and open ended. It has a significant effect on curriculum and instruction in the upper-secondary schools. Even with just this one standardized exam, I heard many Finnish teachers at that level say that they feel pressured and rushed as they try to prepare their students to pass the test. However, this pressure is not related to keeping their jobs or to the school's status or funding; rather, it is related to the hope that they have prepared their students adequately. As part of the ME, all students take an exam in their mother tongue and three subject exams chosen from their second domestic language (Finnish or Swedish), foreign languages, mathematics, and general studies (social and natural sciences).

Mathematics in Finnish upper-secondary schools is taught in seven-to-eight-week courses. The first of these is a course in functions. At the end, students are tested and then move on to the next course, polynomial functions. Course grades are numerical up to 10. Each course has its own small paperback textbook, which the students purchase. Teachers used the textbook extensively during classes, often displaying pages using a document camera. Students can choose between a six- or ten-course sequence in mathematics. If they are interested in opportunities in mathematics or science, they choose the ten-course sequence. The eight additional courses are geometry, analytic geometry, vectors, probability and statistics, differential calculus, root and logarithmic functions, trigonometric functions and sequences, and integral calculus. I observed five advanced ten-course sequence classes, and they were all quite small, having from six to ten students in each. The number of females in those classes was always less than half.

TEACHER PREPARATION

A significant difference in Finland is that mathematics teachers in lower-secondary school (grades 7–9) and upper-secondary school (grades 10–12) must have a master's degree in mathematics, whereas preschool and elementary teachers must earn a master's degree in education. Finns are very proud of their research-based education system. All mathematics teachers whom I met have also studied a science topic such as physics or chemistry in their university work, and they teach both subjects. Teachers in other subjects, such as language, teach two languages, for example; they said it would be more difficult to find a job if you did not have the capability to teach more than one subject.

Teachers typically move with their students so that they have the same students for two to four years, enabling them to get to know one another well. This is especially true in the elementary and

lower-secondary grades. When I told a Finnish teacher of a ninth-grade mathematics class that American teachers quite often teach in just one subject area, such as mathematics, and might teach the same class to the same grade level year after year, she was surprised and thought such a schedule would not be very interesting. She had worked with her ninth graders for three years in both mathematics and chemistry.

Finland has high requirements for entering teacher education programs in universities and, as mentioned before, beginning teachers must earn a master's degree. Finnish teachers' salaries are better than in the United States. They make approximately 92 percent of what other college-educated workers make; in the United States, that figure is less than 60 percent (OECD 2011, p. 15).

Instead of test-based accountability, Finns use the words *responsibility* and *trust*. According to Sahlberg (2011, p. 125), their use of "sample-based testing, thematic assessments, reflective self-evaluations, and emphasis on creative learning have established a culture of mutual trust and respect within the Finnish education system. Before the end of upper-secondary school, or grade 12, no external high-stakes tests are given. There is no inspection of teachers, and only loose external standards steer the schools."

Cooperation among teachers, schools, and communities is encouraged rather than competition. As a result, teaching is a popular career in Finland, and more students apply than are accepted into the programs. In fact, only approximately the top 10 percent of applicants are chosen. Another difference in Finnish teacher preparation is the use of training schools for a portion of student teaching. These schools work closely with the university teacher education departments. The master teachers at the training schools are specially chosen and receive better pay than at other schools.

ESTEEM FOR TEACHERS

Finally, teachers are held in high esteem in Finland. I found classrooms to be casual environments where students called teachers by their first name, but students were almost always respectful and on-task during instruction and independent work. This respect is a result of a number of factors. One teacher told me that during the period after World War II, three pillars of society rebuilt Finland: priests, doctors, and teachers. Society still views teachers and schools as having great value, although some teachers said they feel that this respect is eroding somewhat, especially in larger cities.

The greatest contrast to Finland that I found in Sweden was related to teacher respect, as described by teachers, others I spoke with, and the Swedish press. Every teacher whom I met in Sweden felt that teachers are poorly respected and are blamed for many of Swedish society's problems. The atmosphere of the Swedish classrooms that I observed was different from that of the Finish classrooms. I observed more challenge to teacher authority and more off-task student behavior, especially in lower-secondary classrooms. One Swedish teacher mentioned that he felt that earning respect is a part of Swedish culture; respect is not given just because of one's position.

Swedish teachers also said that government-mandated changes and reforms are too frequent and cause stress and extra work for teachers. In comparison with Finland, it is relatively easy to become a teacher in Sweden (a bachelor's degree is sufficient, as in the United States). In fact, Sweden currently has a shortage of teachers because today teaching is not considered an attractive occupation.

CHANGES COMING

Just days before I left Sweden, I was excited to read an article in the DN.Debatt (debate) page of the Swedish newspaper *Dagens Nyheter* (*Today's News*) dated June 3, 2013. Ten changes that are proposed for Sweden were described by Jan Björklund, education minister; Eva-Lis Sirén, federal chairman of the Lärarförbundet (teachers' union); Bo Jansson, federal chairman of the Teachers' Federation; Anders Knape, chairman of Sweden's municipalities and counties (SKL); and Mikaela Valtersson, chairman of the Independent Schools National Association. Many of these changes are patterned after Finnish ways (I have added "as in Finland" in those cases):

1. Raise teacher salaries (as in Finland).
2. Reduce administration requirements so that teachers are freed to concentrate on planning and implementing instruction.
3. Create an identification and eligibility certificate program, requiring proper training for all who teach a subject (this has already started and will be fully implemented by 2016; for example, it means that only elementary teachers with a specialty in math will be able to teach math).
4. Higher admission requirements will start in 2014 for basic eligibility to enter university education programs (as in Finland).
5. Institute an aptitude test for admission to teacher education, because good theoretical knowledge is not enough. Teachers must be good at communicating and interacting with others (as in Finland).

6. Didactics is an important part of teacher training. Master teachers should be involved in new teacher training (as in Finland).
7. Training schools should be established for all or part of their internship with qualified supervision provided (as in Finland).
8. Monitoring, evaluation, and assessment of student teachers should be improved so that they can grow in their teaching abilities during this time.
9. The first year of teaching is probationary and should be well developed with good mentor support (this has already started).
10. An educational research institute should be established; teaching in the classroom should be research-based to improve quality and outcomes of education in close cooperation with teachers and principals (as in Finland).

Many of the changes proposed in Sweden and the differences that I observed in Finland are not areas that individual teachers can change, but change on a larger scale is already beginning in the United States. Teacher preparation programs such as the University of Indianapolis, the University of South Dakota, Teachers College at Columbia University, and my own institution, Bethel University, have implemented or are developing full-year clinical practice for preservice teachers with specially trained clinical faculty and are raising the requirements for entrance to and completion of teacher preparation programs (Wilson 2013).

Some things that teachers can do are to strive to be worthy of respect by being experts in the mathematics we teach and in best teaching practices, to be willing to share our help and experience by mentoring both experienced teachers and new teachers. We can voice our opinions in favor of changes that will improve mathematics learning for as many students as possible here in the United States.

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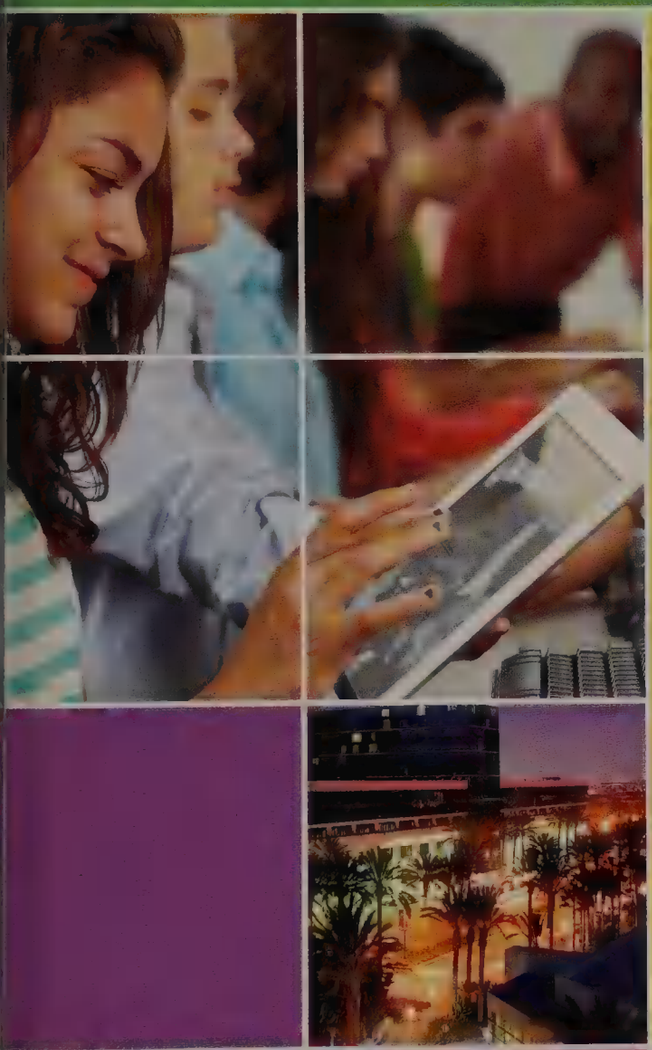

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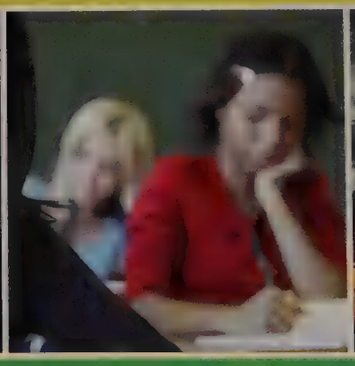

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In the addition problem

$$AH + HA = DAD,$$

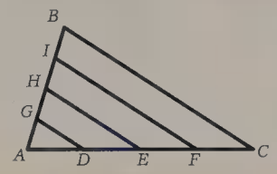
each different letter in the two-digit and three-digit numbers represents a different digit. Find the number represented by DAD.

1

The lengths of the sides of $\triangle ABC$ are $AB = 17$, $BC = 24$, and $AC = 20$. If the lengths $AD = DE = EF = FC$ and

$$\overline{GD} \parallel \overline{HE} \parallel \overline{IF} \parallel \overline{BC},$$

find IF.



2

Two sides of a parallelogram measure 6 cm and 7 cm. Find the lengths of the diagonals if both are integers.

3

All nine entries of the mystical multiplication square are positive integers, but they may not all be different. The product for each row, column, and the two main diagonals is constant. If no two rows or columns contain the same three numbers, find the missing numbers that result in the least product.

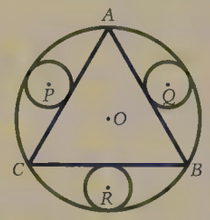
	48	
		3
12		

4

The average length of a toothpick is 5 cm. If a dozen toothpicks are arranged in a straight line with a 2 cm space between them, what is the total expected length of the lineup of toothpicks?

5

Equilateral $\triangle ABC$ is inscribed in circle O, and circles P, Q, and R are tangent to both the triangle and circle O. Find the ratio of the area of $\triangle ABC$ to the sum of the areas of the three small circles.



6

Several cylinders are of different sizes, but for each one the surface area and the volume have the same numerical value. If the radius and height are both integers, what is the largest volume possible?

7

Prove that in a parallelogram the sum of the squares of the diagonals equals the sum of the squares of the four sides.

8

The nonparallel sides of a trapezoid are 13 and 20, a third side is 9, and the trapezoid's height is 12. Find the possible lengths for the fourth side of the trapezoid.

9

Find the sum of the squares of the lengths of the diagonals for each of the two trapezoids in the solution to problem 9.

10

For the trapezoids of the problem for April 10, the sum of the squares of the diagonals is less than the sum of the squares of the four sides. Explain how the differences are related.

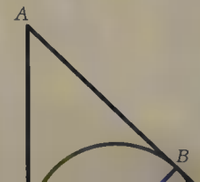
11

The numbers 1, 2, 3, ... 10 are written on ten index cards, one number per card. You shuffle the cards and, without looking at them, randomly select one card and throw it away. If you randomly choose a second card, what is the probability that it will have a prime number written on it?

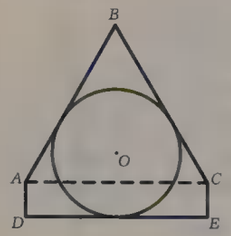
Two concentric circles are drawn so that the smaller circle trisects a chord of the larger circle. The length of the chord equals the sum of the lengths of the two radii. Find the ratio of the smaller radius to the larger radius.

How many three-digit primes have all three digits as different primes?

Semicircle $\overset{\frown}{FBE}$ of circle C is tangent to the hypotenuse AD of right triangle AED at B. If $CE = 60$ and $BD = 63$, find the area of $\triangle AED$.

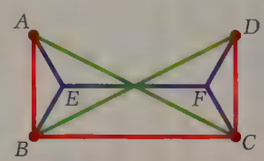


Equilateral triangle ABC sits on side $\overline{AC}$ of rectangle $ACED$. Circle O is tangent to two sides of the triangle and to the base of the rectangle. The length of the radius of the circle is $8\sqrt{3}/3$, and $DE = 12$. Find the area of rectangle $ACED$.



16

Four computers in a room are arranged at the vertices A, B, C, and D of a 6 ft. × 10 ft. rectangle. They are all to be connected with wire. Which of the three possible arrangements shown as red, green, or blue wire in the diagrams requires the least amount of wire?



17

Given $\triangle ADC$ with $AD < AC < DC$, find the probability that the triangle is acute (i.e., that all three angles are acute).

18

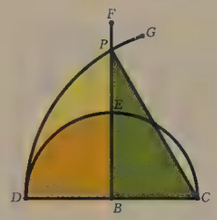
If $9^x - 9^{x-1} = 24$, find $(6x)^x$.

19

The sum of the positive divisors of 120 is 360. Find the sum of the reciprocals of all the positive divisors of 120.

20

The yellow region in the diagram is bounded by circles centered at B and C and the line FB. Assume that $DB = 1$. If a circle is inscribed in the yellow region (internally tangent to $\overset{\frown}{PD}$, externally tangent to $\overset{\frown}{ED}$ and tangent to $\overline{PE}$), find its radius.



21

Two congruent squares overlap to form a regular octagon and an equilateral hexadecagon (16 sides). Find the ratio of the area of the hexadecagon to the area of the octagon.

22

In circle O the length of the radius is x, and in circle P the length of the radius is y. The common internal tangent is $\overline{RQ}$, and the line of centers is $\overline{OP}$. Prove the following:

$$RQ^2 + (x+y)^2 = OP^2$$

23

In $\triangle ABC$, the bisector of $\angle A$ intersects $\overline{BC}$ at D. If $AB = 8$, $AC = 15$, and $m\angle A = 60°$, find BD.

24

The angle bisector of $\angle A$ in $\triangle ABC$ divides $\overline{BC}$ into two segments of lengths 4 and 6. If the length of the angle bisector is 8, compute the product $AB \cdot AC$.

25

Simplify the following fraction:

$$\frac{(1+2+3+\cdots+19)\cdot 20}{(21+22+23+\cdots+39)\cdot 40}$$

26

Two congruent equilateral triangles overlap to form a regular hexagon and an equilateral dodecagon (12 sides). Find the ratio of the area of the dodecagon to the area of the hexagon.

27

The circumference of a large circle is 2 cm greater than the circumference of a small circle. How much bigger is the radius of the large circle than the radius of the small circle?

28

Prove the general statement introduced numerically in problem 26. Show that for all integers n,

$$\frac{(1+2+3+\cdots+(2n-1))\cdot 2n}{((2n+1)+(2n+2)+(2n+3)+\cdots+(4n-1))\cdot(4n)} = \frac{1}{6}.$$

29

The number 2015 can be written as the sum of two or more consecutive positive integers in seven different ways. Find the total of the smallest integers from each of the seven ways.

30

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SOLUTIONS to calendar

1. 121. The sum is a three-digit number, so D must equal 1. Since $D = 1$, $H + A$ equals 1 or 11. We can reject 1 because it would require H or A to be 0. For any digits H and A with a sum of 11, the final sum for DAD will be 121.

2. 18. The parallel lines are equally spaced, so $AF = (3/4)AC$. Since $\triangle AFI \sim \triangle ACB$, corresponding sides of these triangles will be proportional. Thus, $FI = (3/4)BC = (3/4)(24) = 18$.

3. 7 cm and 11 cm. The most direct approach is to apply the theorem that the sum of the squares of the diagonals in a parallelogram equals the sum of the squares of the four sides. Therefore, if the diagonal lengths are a and b, then $a^2 + b^2 = 2(6^2 + 7^2) = 170$. Two pairs of integer lengths (1, 13) and (7, 11) satisfy this equation. The triangle inequality theorem requires the diagonals of the parallelogram to be between $7 - 6 = 1$ and $7 + 6 = 13$. Thus, we must reject the first pair of values.

***Alternate solution*:** Apply the law of cosines twice to get $a^2 = 6^2 + 7^2 - 2(6) \cdot (7)(\cos A)$ and $b^2 = 6^2 + 7^2 - 2(6)(7) \cdot (\cos(180° - A))$. Using $\cos(180° - A) = -\cos A$, add the two equations to arrive at the same relationship as above: $a^2 + b^2 = 2(6^2 + 7^2) = 170$.

4. The rows are {6, 48, 48}, {192, 24, 3}, and {12, 12, 96}. Let the missing entries in the top row be a and b. The product of every row, column, and diagonal is then $48ab$. For the diagonal from lower left to upper right to have a product of $48ab$, the middle square must be $4a$. Complete the middle row with $4b$. The bottom row then contains $b/4$ and $16a$, according to the product in each column. The diagonal from upper left to lower right gives $64a^3 = 48ab$, so $4a^2 = 3b \rightarrow b = (2a)^2/3$. Because we want the least product, try $a = 3$; then $b = 12$, and the result has all rows containing the same three numbers. If $a = 6$, then $b = 48$, and the rows are {6, 48, 48}, {192, 24, 3}, and {12, 12, 96}; these satisfy the conditions of the question.

a	48	b
$4b$	$4a$	3
12	$b/4$	$16a$

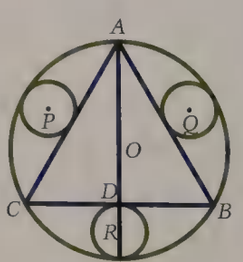
5. 82 cm. We have 12 toothpicks but only 11 spaces. Thus, $12(5) + 11(2) = 82$ cm.

6. $4\sqrt{3} : \pi$. Draw the diameter of circle O through vertex A so that the diameter intersects CB at D. Since $\overline{AD}$ is a median of equilateral triangle ABC, the ratio $AO/OD = 2/1$, meaning that $OD = AO/2$. It follows that the diameter of circle R is also $AO/2$, so its radius is $AO/4$. Let $AO = 4$; then the area of each small circle $\pi(1)^2 = \pi$. The altitude of the triangle is 6, so a side of the triangle is $4\sqrt{3}$. The area of $\triangle ABC$ is

$$\frac{\sqrt{3}}{4}\left(4\sqrt{3}\right)^2 = 12\sqrt{3}.$$

The sum of the areas of the three small circles is 3π. The requested ratio is

$$\frac{12\sqrt{3}}{3\pi} = \frac{4\sqrt{3}}{\pi}.$$

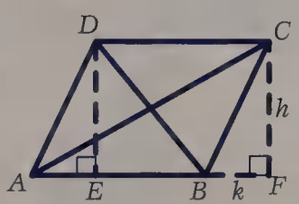


The Editorial Panel of *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

Department editors
Margaret Coffey, *Margaret.Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312; and* ***Art Kalish,*** *artkalish@verizon.net, Director of the Institute of MERIT at SUNY College at Old Westbury*

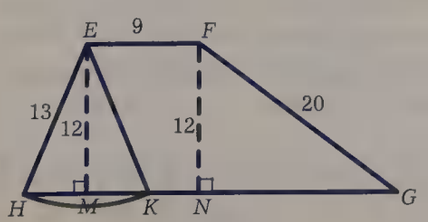
7. 108π. The volume of each cylinder is $\pi r^2 h$, and the surface area of each is $2\pi r^2 + 2\pi rh$. Set these equal and solve for h: $h = (2r)/(r - 2) = 2 + 4/(r - 2)$. Because h must be an integer, $r - 2$ must be a factor of 4—that is, ± 1, ± 2, or ± 4. Checking each of these so that r and h are both positive integers results in three solutions. Writing the ordered pair (r, h), we get (3, 6), (4, 4), and (6, 3). The volumes of the corresponding three cylinders are 54π, 64π, and 108π, respectively.

8. Label the parallelogram $ABCD$, draw altitudes from D and C, and label their intersections E and F, as shown in the figure. Let the length of the altitudes be h, and let $AE = BF = k$. Let $AB = CD = p$, and let $AD = CB = q$. We find that $BE = p - k$ and that $AF = p + k$. Apply the Pythagorean theorem in ΔBDE to get $h^2 + (p - k)^2 = BD^2$. Then apply the theorem in $\triangle ACF$ to get $h^2 + (p + k)^2 = AC^2$. Add these results together to get $BD^2 + AC^2 = h^2 + (p - k)^2 + h^2 + (p + k)^2 = 2h^2 + 2k^2 + 2p^2$. Note that in $\triangle BCF$, $h^2 + k^2 = CB^2 = q^2$. Thus, $BD^2 + AC^2 = 2q^2 + 2p^2$.

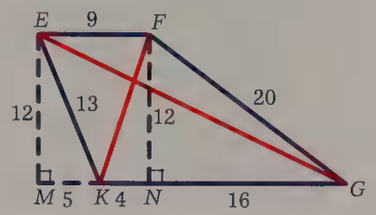


Alternate solution: Apply the law of cosines in $\triangle BAD$ and $\triangle ABC$. Notice that $\angle BAD$ and $\angle ABC$ are supplementary, so the cosines of the angles are opposites—that is, $\cos(ABC) = -\cos(BAD)$. Thus, $BD^2 = p^2 + q^2 - 2pq\cos(BAD)$, and $AC^2 = p^2 + q^2 + 2pq\cos(BAD)$, so $BD^2 + AC^2 = 2q^2 + 2p^2$.

9. 20 and 30. The two possible trapezoids, $EFGH$ and $EFGK$, can be seen in the diagram. Apply the Pythagorean theorem to find that $HM = MK = 5$ and $GN = 16$. Therefore, the fourth side is either $GH = GN + NM + HM = 16 + 9 + 5 = 30$ or $GK = GN + NM - MK = 16 + 9 - 5 = 20$.

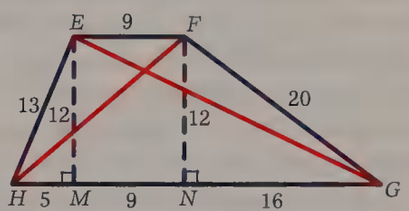


10. 929 and 1109. When the fourth side is 20, the sum of the squares of the diagonals is 929. When the fourth side is 30, the sum of the squares of the diagonals is 1109. Consider the measurements found in the two diagrams and apply the Pythagorean theorem. In **figure a**, $GE^2 = 12^2 + 25^2 = 144 + 625 = 769$ and $FK^2 = 4^2 + 12^2 = 16 + 144 = 160$. The sum is $769 + 160 = 929$.



(a)

In **figure b**, $GE^2 = 769$ and $FH^2 = 14^2 + 12^2 = 196 + 144 = 340$. In this case, the sum is $769 + 340 = 1109$.



(b)

11. The differences are $121 = (16 - 5)^2$ and $441 = (16 + 5)^2$. In **figure a** in solution 10, the sum of the squares of the four sides is $13^2 + 9^2 + 20^2 + 20^2 = 1050$ and the sum of the squares of the diagonals is 929. The difference, 121 is the square of $11 = 16 - 5$. In **figure b** in solution 10, the sum of the squares of the four sides is $13^2 + 9^2 + 20^2 + 30^2 = 1550$, and the sum of the squares of the diagonals is 1109. The difference, 441, is the square of $21 = 16 + 5$.

These values might give us a hint as to how to start the proof of a general theorem for variables defined in the diagram:

$$(r^2 + q^2 + s^2 + p^2) - (v^2 + w^2) = (AG \cdot \operatorname{sgn}(AB - BG) + BE \cdot \operatorname{sgn}(BA - AE))^2$$

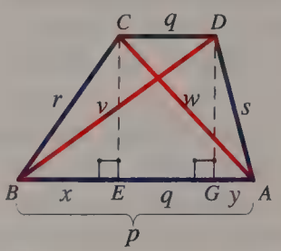
where

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -1 & \text{when } x < 0 \end{cases}$$

defines the signum function.

In the case of a parallelogram, we have $BE = AG$, and one instance of the signum function equals 1 whereas the other equals -1, so the result is an equality as in the problem of April 8.

Another interesting result based on this diagram follows: For a trapezoid with fixed-length diagonals, the minimum perimeter occurs when the trapezoid is a parallelogram, and the maximum perimeter occurs when one side of the trapezoid has length zero—that is, when the quadrilateral turns into a triangle (the diagonals become the sides of the triangle).



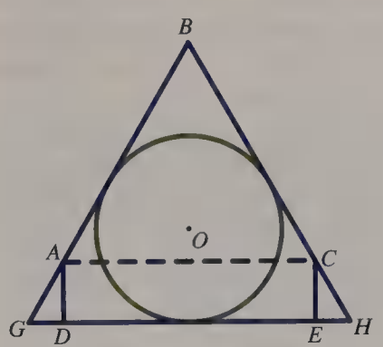
12. 4/10. The prime numbers in the set 1 through 10 are 2, 3, 5, and 7. If no card was thrown away, the probability of a chosen card displaying a prime would be 4/10. Each card in the deck has the same probability of containing a prime number. Throwing away a card will have no effect. It would be the same as finding the probability that the second card from the top has a prime number written on it.

13. 7/11. Let O be the center of the two circles with radii $a < b$. The chord is trisected; let the length of each of the three segments be $2x$; then from the problem statement we have $a + b = 6x$. Let the distance from O to the chord be y. Since a radius perpendicular to a chord bisects the chord, $HF = x$. Apply the Pythagorean theorem in $\triangle OHF$ and again in $\triangle OHE$. Thus, $y^2 = OF^2 - HF^2$, and $y^2 = OE^2 - HE^2$. Set these expressions equal: $OF^2 - HF^2 = OE^2 - HE^2 \rightarrow a^2 - x^2 = b^2 - (3x)^2 \rightarrow 8x^2 = b^2 - a^2 = (b - a)(b + a) = 6x(b - a)$. Thus, $b - a = 4x/3$. Add to this $b + a = 6x$; then $2b = 22x/3$ and $b = 11x/3$. Substitute to find $a = 7x/3$; so $a/b = 7/11$.

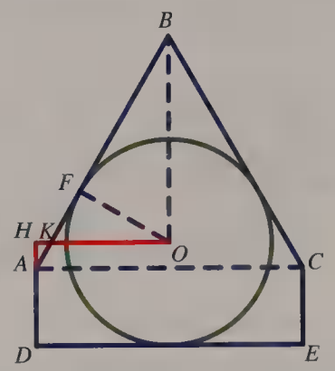
14. Two: 523 and 257. To begin, list the prime-number digits: 2, 3, 5, and 7. A three-digit number ending in 2 or 5 will not be prime. Therefore, we are looking for numbers that end in 3 or 7. If the units digit is 3, the other two digits cannot be 2 and 7 or 5 and 7 because the sum of all three digits will be a multiple of 3 and therefore the three-digit number will be a multiple of 3. The only two possible numbers ending in 3 are $253 = 11(23)$ and 523, which is prime. Apply the same reasoning to numbers ending in 7. The other two digits must be 2 and 5. The number 257 is prime, but $527 = 17(31)$.

15. 10,290 units^2. Since B is a point of tangency, radius $BC = CE = 60$, and $\angle CBD$ is a right angle. Applying the Pythagorean theorem in $\triangle DBC$ gives $CD^2 = 60^2 + 63^2$; then $CD = 87$ and $DE = 87 + 60 = 147$. Since $\triangle DBC \sim \triangle DEA$, we have $AE/147 = 60/63$, so $AE = 140$. The area of $\triangle AED = (140 \cdot 147)/2 = 10{,}290$.

16. $24\sqrt{3}$. Extend the sides of $\triangle ABC$ until they intersect the line containing $\overline{DE}$; call the intersections G and H. Then $\triangle BGH$ is also equilateral. Since the radius of the circle inscribed in $\triangle BGH$ is $8\sqrt{3}/3$, the altitude of the triangle is $8\sqrt{3}$, and each side of the triangle is 16. Therefore, $GD = HE = 2$ because $DE = 12$. With $m\angle G = 60°$, we know that $AD = 2\sqrt{3}$. Thus, the area of the rectangle is $12(2\sqrt{3}) = 24\sqrt{3}$.



Alternate solution: Create $\triangle AKH$ by extending $\overline{DA}$ through A until it meets the line parallel to $\overline{DE}$ drawn through point O. This new triangle as well as $\triangle BOF$, $\triangle OKF$, and $\triangle BKO$ are each 30-60-90° triangles with side ratios of $1 : \sqrt{3} : 2$. With $OF = 8\sqrt{3}/3$, we have $BF = 8$ and $FK = 8/3$. With $AB = 12$, we find $AK = 12 - (BF + FK) = 12 - (8 + 8/3) = 4/3$. Then $AH = (\sqrt{3}/2) = 2\sqrt{3}/3$. Finally, HD is the length of a radius of the circle, or $8\sqrt{3}/3$, so $AD = HD - AH = 8\sqrt{3}/3 - 2\sqrt{3}/3 = 6\sqrt{3}/3 = 2\sqrt{3}$. The area of the rectangle equals $12(2\sqrt{3}) = 24\sqrt{3}$.

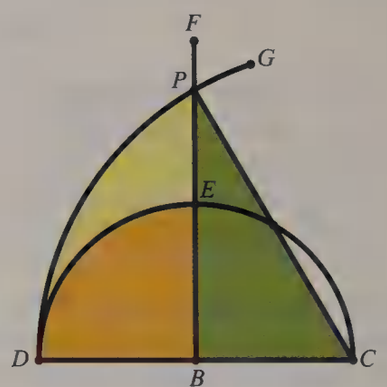


17. The configuration shown in blue uses the least wire, provided the angles at vertex E and vertex F are each 120°. The total length of the red wire is $6 + 10 + 6 = 22$ ft. Finding the total length of the green wire requires using the Pythagorean theorem to determine the length of one diagonal segment as $\sqrt{136}$. Thus, the total length is $2\sqrt{136} \approx 23.32$ ft. If the angles made by the blue wire at vertices E and F are each 120°, then $AE = BE = DF = CF = 2\sqrt{3}$ and $EF = 10 - 2\sqrt{3}$. Therefore, the total length of wire needed is $4(2\sqrt{3}) + (10 - 2\sqrt{3}) = 10 + 6\sqrt{3} \approx 20.39$ ft. Readers who are interested in why the 120° angle minimizes the distance are encouraged to look up the concept of a Steiner point, named for Jakob Steiner, a nineteenth-century mathematician who made many contributions to geometry.

18. $\dfrac{5\pi - 6\sqrt{3}}{8\pi - 6\sqrt{3}} \approx 0.361$. Draw the longest side $\overline{DC}$. Since $AD < AC$, point A must be to the left of the perpendicular bisector of $\overline{DC}$ (see diagram in col. 3). If A is on semicircle $\overset{\frown}{CED}$ centered at B, the measure of the angle at A would be 90°. If A were selected within the semicircle (in the orange region), $\angle A$ would be obtuse. Thus, A must be chosen above the semicircle so that the triangle is acute. So that $AC > DC$, point A must be below $\overset{\frown}{GPD}$ centered at C. Therefore, A must be in the yellow region bounded by $\overset{\frown}{PD}$, $\overset{\frown}{DE}$, and $\overset{\frown}{FB}$. The probability that point A resides in the defined region forming an acute triangle can be found by dividing the area of that region by the area of the half segment PDB. Without loss of generality, we can let $DB = BC = 1$, which makes $PC = 2$, and we can determine that $m\angle PCB = 60°$. The area of sector $CPD = 2^2\pi/6 = 2\pi/3$. The area of $\triangle PBC = 1 \cdot \sqrt{3}/2 = \sqrt{3}/2$. The area of sector $BED = 1^2\pi/4 = \pi/4$. The ratio we wish to calculate is

$$\frac{\text{area of sector } CPD - \left(\text{area of } \triangle PBC + \text{sector } BED\right)}{\text{area of sector } CPD - \text{area of } \triangle PBC} = \frac{2\pi/3 - \left(\sqrt{3}/2 + \pi/4\right)}{2\pi/3 - \sqrt{3}/2} = \frac{5\pi - 6\sqrt{3}}{8\pi - 6\sqrt{3}}.$$

(For a related problem, see "Reconciling Representations," by William Zahner and Nick Dent, *MT* Dec. 2014/Jan. 2015, vol. 108, no. 5, pp. 344–54.)

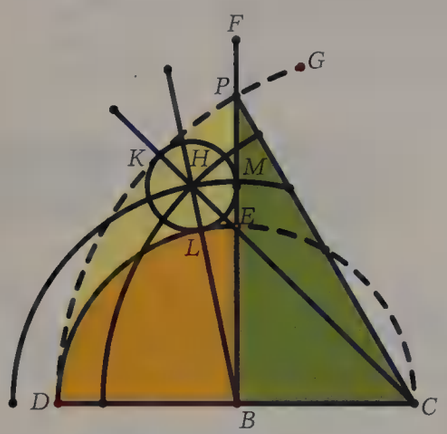


19. 27. Since $9^x - 9^{x-1} = 9^{x-1}(9 - 1) = 24$, we have $9^{x-1} = 3 \rightarrow x = 3/2$. Therefore, $(6x)^x = 9^{3/2} = 3^3 = 27$.

20. 3. Consider a simpler problem, such as to find the sum of the reciprocals of the divisors of 6. The divisors are 1, 2, 3, and 6. The sum that we seek is $1/1 + 1/2 + 1/3 + 1/6$. The common denominator is 6, so find the sum of $6/6 + 3/6 + 2/6 + 1/6 = (6 + 3 + 2 + 1)/6$. This is the sum of the divisors of 6 divided by 6. Similar reasoning shows that the answer to the original question is the sum of the divisors of 120 divided by 120: $360/120 = 3$.

21. 1/4. Let the center of the inscribed circle be H. Draw a ray from C and a ray

from B so that they will intersect at H. Let the radius of the inscribed circle be m. Place the diagram on a coordinate plane with D (0, 0), B (1, 0), and C (2, 0). Then circle B with radius BH has equation $(x-1)^2+y^2=(1+m)^2$, and circle C with radius CH has equation $(x-2)^2+y^2=(2-m)^2$. The intersection of these circles can be found by solving the system of equations, resulting in $x=3m$. In addition, we know that the distance from H to $\overline{PB}$ is m, so H has an x-coordinate of $1-m$. Set these two expressions for x equal to get $3m=1-m$. Thus, $m=1/4$, the radius of the inscribed circle.



22. $\sqrt{2}$. The hexadecagon is formed by the square plus four triangular regions congruent to triangle EFG, and the octagon is formed by the square minus four triangular regions. Let $AF=FG=GE=BE=x$. Use the Pythagorean theorem or the properties of a right isosceles triangle to find that $FE=x\sqrt{2}$, so each square has side length $x+x\sqrt{2}+x$. The area of the square is

$$\left(x\left(2+\sqrt{2}\right)\right)^2=x^2\left(6+4\sqrt{2}\right),$$

and the area of each right isosceles triangle is $(GF \cdot GE)/2=x^2/2$. The area of the hexadecagon is

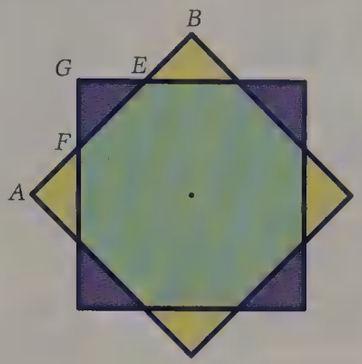
$$x^2\left(6+4\sqrt{2}\right)+4\left(x^2/2\right)=x^2\left(8+4\sqrt{2}\right).$$

The area of the octagon is

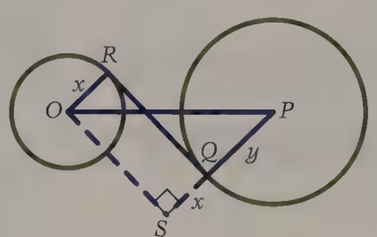
$$x^2\left(6+4\sqrt{2}\right)-4\left(x^2/2\right)=x^2\left(4+4\sqrt{2}\right).$$

The ratio of the areas is

$$\frac{x^2\left(8+4\sqrt{2}\right)}{x^2\left(4+4\sqrt{2}\right)}=\sqrt{2}.$$



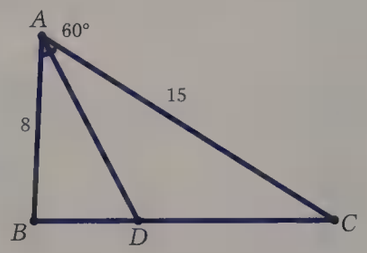
23. Construct a line parallel to $\overline{RQ}$ through point O. Extend $\overline{PQ}$ until it meets the line just constructed and label the intersection S. A tangent is perpendicular to a radius at the point of tangency, so $\overline{PQ} \perp \overline{RQ}$. Since $\overline{RQ} \parallel \overline{OS}$, $\overline{PS} \perp \overline{OS}$. Polygon $ORQS$ is a rectangle because $\angle R$, $\angle RQS$, and $\angle S$ are all right angles, so $QS=x$ and $OS=RQ$. Apply the Pythagorean theorem to $\triangle OPS$ to get $OS^2+(x+y)^2=OP^2$. Replace OS with RQ to complete the proof.



24. $104/23 \approx 4.5217$. Find BC by applying the law of cosines in $\triangle ABC$. Then

$$BC^2=AB^2+AC^2-2(AB)(AC)\cos(A)$$
$$BC^2=8^2+15^2-2(8)(15)\cos(60°)$$
$$=64+225-120=169$$
$$BC=13.$$

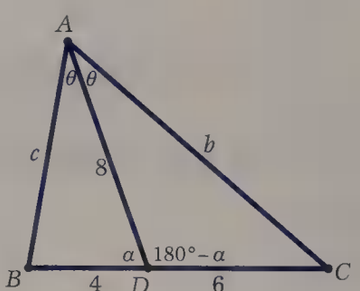
The bisector of an angle divides the opposite side in the same ratio as the sides adjacent to the angle, so $BD/DC=AB/AC=8/15$. Taking reciprocals, we have $DC/BD=15/8$. In a proportion, we can add the numerators to the denominators to get another proportion. (Adding 1 to each side of the equation gives $(DC+BD)/BD=(15+8)/8$. Taking reciprocals again gives $BD/(DC+BD)=8/(15+8)$.) Thus, $BD/13=8/23 \rightarrow BD=8(13)/23=104/23$.



25. 88. Apply the theorem that the bisector of an angle of a triangle divides the opposite side into the same ratio as the sides adjacent to the bisected angle. Let $c=4x$ and $b=6x$. Use the law of cosines on the two smaller triangles to find that $(4x)^2=4^2+8^2-2(4)(8)\cos(\alpha)$ and $(6x)^2=6^2+8^2-2(6)(8)\cos(180°-\alpha)$. Since $\cos(180°-\alpha)=-\cos(\alpha)$, we now have $16x^2=80-64\cos(\alpha)$ and $36x^2=100+96\cos(\alpha)$. Solve each equation for $\cos(\alpha)$ and set the results equal:

$$\frac{80-16x^2}{64}=\frac{36x^2-100}{96}$$

Thus, $x^2=11/3$. The product of the lengths of the two sides is $24x^2=24(11/3)=88$.



Alternate solution: A theorem states that in a triangle with an angle bisector drawn to one side, the product of the two sides adjacent to the bisected angle equals the sum of the squared length of the angle bisector and the product of the two parts of the divided side. Specifically, for the diagram above, $AB \cdot AC=AD^2+BD \cdot DC=8^2+6 \cdot 4=64+24=88$.

Proof of theorem: Let $AB=c$, $AC=b$, $AD=p$, $BD=q$, and $DC=r$. We wish to show that $bc=p^2+qr$. Construct the circle circumscribed about $\triangle ABC$. Extend the angle bisector until it meets the circle and label that point E. Draw $\overline{CE}$. Notice that $\angle ABC \cong \angle AEC$ since they

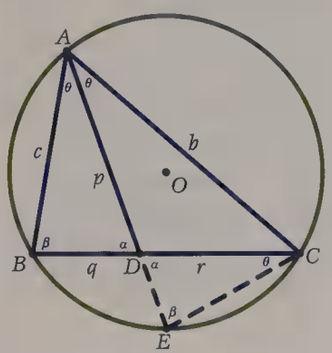
are inscribed angles intercepting the same arc. It follows that $\triangle ABD \sim \triangle AEC \sim \triangle CED$. From these similar triangles, we have

$$\frac{AB}{AD} = \frac{AE}{AC} \rightarrow \frac{c}{p} = \frac{AE}{b}$$

and

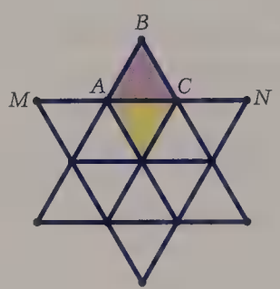
$$\frac{AD}{BD} = \frac{CD}{DE} \rightarrow \frac{p}{q} = \frac{r}{DE}.$$

Therefore, $bc = p(AE)$ and $p(DE) = qr$. But $AE = p + DE$, so by substitution, $bc = p(p + DE) = p^2 + p(DE) = p^2 + qr$.



26. 1/6. The sum in both numerator and denominator can be found using the formula for an arithmetic series, $S_n = (a_1 + a_n)(n/2)$. Thus, $1 + 2 + 3 + \cdots + 19 = (1 + 19)(19/2) = 190$ and $21 + 22 + 23 + \cdots + 39 = (21 + 39)(19/2) = 570$. Then, $190/570 = 1/3$ is multiplied by $20/40 = 1/2$, to get $(1/3)(1/2) = 1/6$.

27. 2:1. The original equilateral triangle comprises nine smaller equilateral triangles, each congruent to triangle ABC in the diagram. The hexagon is formed from six such triangles. The dodecagon is formed from twelve such triangles. Thus, the ratio of the area of the dodecagon to the area of the hexagon is 12:6 = 2:1.



28. $1/\pi$. Let the radius of the smaller circle be r; then its circumference will be $2\pi r$. Let the circumference of the larger circle be R; then its circumference is $2\pi R = 2\pi r + 2$. Thus, $R = (2\pi r + 2)/(2\pi) = r + 1/\pi$. Hence, the larger radius is $1/\pi$ greater than the smaller radius. Notice that this difference does not depend on the radii.

29. Apply the general formula for the sum of an arithmetic sequence as defined in the solution to 26:

$$\frac{\left(1+2+3+\cdots+\left(2n-1\right)\right)\cdot 2n}{\left(\begin{matrix}\left(2n+1\right)+\left(2n+2\right)\\+\left(2n+3\right)+\cdots+\left(4n-1\right)\end{matrix}\right)\cdot\left(4n\right)}$$

$$= \frac{\left(1+\left(2n-1\right)\right)\frac{2n-1}{2}\cdot 2n}{\left(\left(2n+1\right)+\left(4n-1\right)\right)\frac{2n-1}{2}\cdot 4n}$$

$$= \frac{2n\cdot 2}{6n\cdot 4} = \frac{1}{6}$$

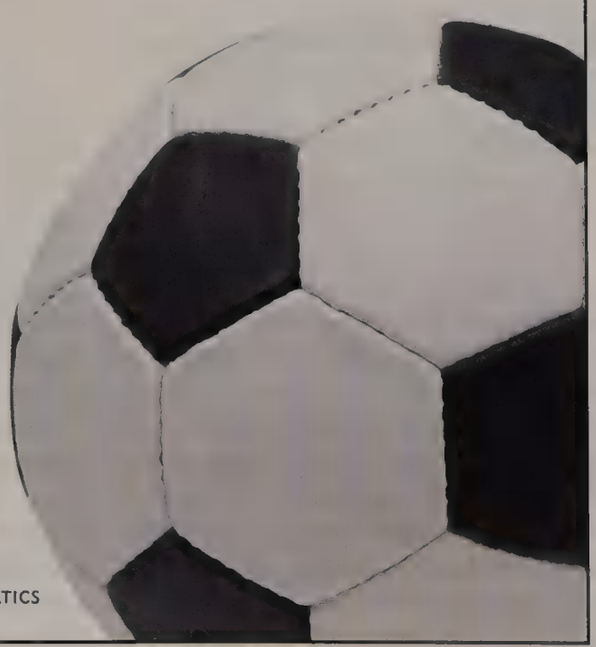
30. 1871. The formula for the sum of a sequence of consecutive integers is $S = (a_1 + a_n)(n/2)$. We want the sum to be 2015, and 2 times 2015 is 4030, so we need to find the divisors of 4030: 2, 5, 10, 13, 26, 31, 62, 65, 130, 155, 310, 403, 806, and 2015. These are the only possible values for n, the number of terms in the series. Because $n \le a_n \le a_1 + a_n$, we only need to consider n less than the square root of 4030. For each possible value of $n \le 62$, we can write $a_n = a_1 + (n - 1)$, and then solve $4030 = (2a_1 + n - 1)n$ to find the starting term, a_1, of a series whose sum is 2015.

n	2	5	10	13	26	31	62
a_1	1007	401	197	149	65	50	2

The total of these seven smallest terms is $1007 + 401 + 197 + 149 + 65 + 50 + 2 = 1871$.

I ♥ spherical analogs of truncated icosahedrons.

MATHEMATICS IS ALL AROUND US.



NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

2016 Focus Issue

Teaching Mathematics Online

Today's students live in a world of computers, tablets, smartphones, and cloud computing. The pace of innovation continues to accelerate. How can mathematics teachers capitalize on these trends for the good of our students? The Editorial Panel of *Mathematics Teacher* invites teachers, teacher educators, education researchers, and others to share their experiences in online mathematics education. Submissions that highlight how online technologies can support NCTM's Equity Principle are especially encouraged. In some cases, print alone will not adequately capture the total impact of the article; in such cases, having active links or other dynamic resources will be optimal.

All possible innovative uses of online resources cannot be enumerated here, but the following questions will get the conversation started.

OPPORTUNITY

How can online technologies—

- be leveraged for mathematics learning?
- provide for student collaboration across distances?
- improve professional development opportunities?
- affect the achievement gaps between different groups of students?

IMPLEMENTATION

How do you—

- remain connected to your students when teaching online?
- use specific online tools to help students learn?
- balance and complement traditional classroom formats with online technologies?
- compare the effectiveness of different methods for student learning?

These are just a few of our ideas; we want to know what you think. We particularly encourage teachers to share their own favorite online lessons.

Please submit manuscripts and Web materials at **mt.msubmit.net** by **May 1, 2015**. Be sure to enter the call's title (Teaching Mathematics Online) in the Department/Calls field. No author identification should appear in the text of the manuscript. Manuscript guidelines are available at www.nctm.org/publications/content.aspx?id=22602. If you have ideas related to this topic and wish to discuss them before sending a manuscript or need details about restrictions on online resources as part of submissions, please contact Tara Slesar (tslesar@nctm.org).

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SMARTER COOKIES

An inquiry-based project to examine statistical claims encourages students to become more savvy media consumers.

Jeremy F. Strayer and Michael Todd Edwards

A news story claiming that the cream filling of a popular "double" cream sandwich cookie is not really "double" went viral in fall 2013. A high school mathematics teacher posted a blog entry describing how he and his students measured 20 cookies, analyzed the data, and concluded that the double cream cookies had only 1.86 times the filling of the regular variety (Anderson 2013). A media frenzy ensued. Countless blog sites and large media outlets—including CNN, *The Huffington Post*, ABC, *Time*, and *The Atlantic*—picked up the story, leading many to conclude that the double cream cookies "aren't all they're stuffed up to be" (Perreira and Payne 2013).

The news story broke while we were teaching statistics topics in entry-level undergraduate courses—an introductory statistics course and a methods course for secondary school mathematics teachers. As mathematics educators, we found the report both encouraging and troubling. We are genuinely excited whenever we see real-world applications play a prominent role in school classrooms. Nevertheless, investigative reports highlighting the use of very small, nonrepresentative samples and dubious data collection strategies are disturbing; they reinforce statistical misconceptions and misunderstandings.

In this sense, the story was immediately compelling and timely; it provided us with a vehicle for engaging students in a meaningful investigation of statistical claims in the popular media. Within a few weeks—and after many hours of collaborative planning—we developed a project that encouraged students to implement statistical experiments of their own design while confronting misunderstandings perpetuated by the news story. Here we describe that project—the Cookie Conundrum—through displays and discussions of our students' work. Although we conceived the Cookie Conundrum as one for use with undergraduates or preservice teachers, the project could certainly be

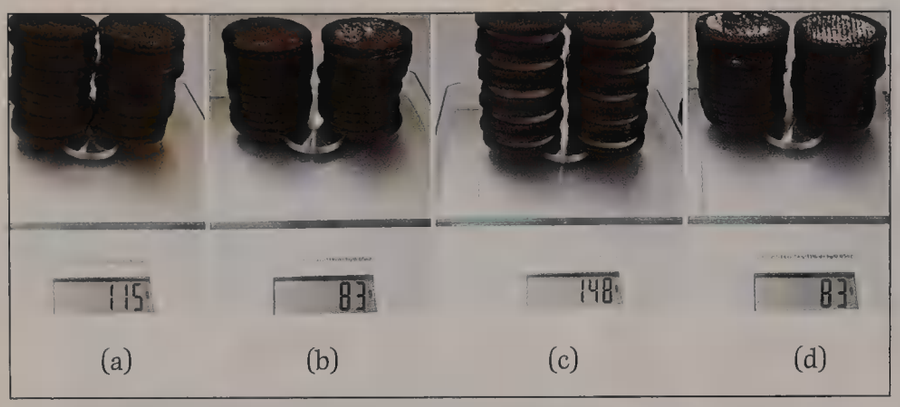

Fig. 1 Students weighed 10 regular cookies (115 g) (**a**), biscuits from the same 10 regular cookies (**b**), 10 double-cream cookies (148 g) (**c**), and biscuits from the same 10 double-cream cookies (**d**).

Fig. 2 Students scraped cream off the biscuit with a driver's license.

adapted for use with mathematics students in the middle school or secondary school grades.

According to the American Statistical Association's *GAISE* report (Franklin et al. 2007), students need repeated opportunities to engage in each of the four components of statistical investigations when learning statistics: (1) formulating questions, (2) collecting data, (3) analyzing data, and (4) interpreting results. We explicitly designed the Cookie Conundrum to engage students in all four GAISE components. In particular, we describe ways in which students formulated their own questions and designed their own research methods to answer their own questions, using appropriate statistical methods to critically examine unfounded claims involving double-cream cookies.

The Cookie Conundrum project consists of three phases. We conceived phase 1 and phase 2 to be completed in approximately one class meeting each, with phase 3 requiring several class meetings. However, we recognize that timing will vary according to the students' prior experiences and content knowledge.

- *Phase 1* (*Replication*): First, students are introduced to the original study. After watching a CNN video highlighting the basics of the news story, small groups replicate the data collection (GAISE component 2) and preliminary data analysis (GAISE component 3), gaining a more detailed understanding of the methods used as they produce comparison data.
- *Phase 2* (*Critique*): Students consider the original cookie investigation and ensuing media coverage more critically. First, they formulate their own research questions (GAISE component 1). Next, they critique the original data collection methods (GAISE component 2) and brainstorm alternatives.
- *Phase 3* (*Reformulation*): Last, students analyze and interpret a revised data set (GAISE components 3 and 4) to answer their own cookie questions.

Next we describe challenges and opportunities that we observed with our students as they engaged in each of the three phases of the project.

PHASE 1: REPLICATING THE ORIGINAL STUDY

We launch the project by showing students a CNN video (http://bit.ly/cnn_cookies) that describes the background of the cookie experiment and the ensuing controversy it prompted. Following the method reported in the video and the teacher's original blog post (Anderson 2013), our students weighed ten regular and ten double-cream cookies, with and without cream, as illustrated in **figure 1**.

For each type of cookie, small teams of three to four students separated cream from biscuits, simultaneously twisting the top and bottom in opposite

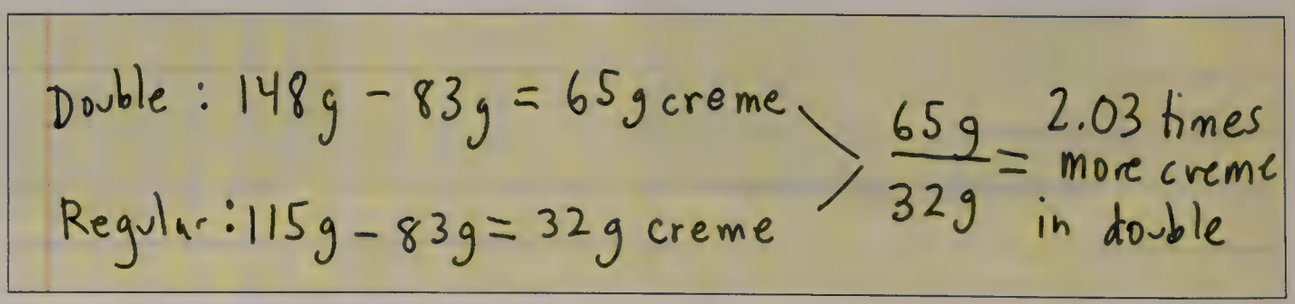

Fig. 3 A student calculated a cream ratio for regular and double-cream sandwich cookies.

directions. This approach left all the cream on one biscuit. As **figure 2** suggests, teams scraped as much of the remaining cream off the bottom biscuit as possible using readily available materials.

Next, teams subtracted the mass of 10 cookies without cream (i.e., the biscuits) from the mass of the same cookies with cream. This calculation provided an estimate of the mass of the cream, which students used to calculate the cream ratio for 10-cookie samples. Student work is shown in **figure 3**.

Similar calculations by five other teams are provided in **table 1**. As the table indicates, the teams generated conflicting results. Some cream ratios were larger than 2, whereas others were less. This variation, coupled with remarks from the cookie manufacturer stating that "our recipe . . . has double the stuff, or cream filling, when compared with our . . . original" (Perreira and Payne 2013), brought into question both the design of the reported investigation as a statistical study as well as the subsequent media coverage. Confronted with additional data, many students questioned whether newsroom editors and correspondents had taken the time to perform the cookie experiment before reporting their findings.

A prospective high school teacher in our methods course shared the following observations: "It is absolutely amazing to me that the cookie story went viral based on data obtained from 20 cookies in two bags. In 15 minutes, we obtained results that contradicted details presented in the online article. Personally, it's difficult for me to believe that others would report that double cookies have less than twice the icing using data collected this way" (Student A, personal communication, Nov. 21, 2013).

The simple act of repeating the data collection (GAISE component 2) and mathematical analysis (GAISE component 3) allowed our students to experience natural variation that exists in repeated processes, providing an important point of reference as they developed statistical data analysis methods in subsequent phases of the project.

PHASE 2: CRITIQUING THE ORIGINAL STUDY

In the next phase, we asked students to reflect on weaknesses of the original cookie investigation as they critiqued the data collection procedures from phase 1 and formulated their own research questions (GAISE component 1). To help focus classroom conversations and generate discussion, we divided students into a number of small "question" and "data" groups, each consisting of three to four students. Individual question groups were asked to state the aim of the original study as a one-sentence research question. Data groups brainstormed concerns related to data collection procedures. After some time, the small groups came together, forming large question and data groups. The larger groups shared initial ideas, consolidating their findings as they prepared a short presentation for the whole class. Below we provide observations that students shared in their whole-group presentations.

Question Formulation

The following research questions were generated by question groups in our classes.

1. Does a double-cream cookie have double the cream of a regular cookie?
2. Do double-cream cookies have double the cream when compared with regular cookies?
3. Is the average amount of cream in double-cream cookies equal to 2 times the average amount of cream in the regular cookies?

These three questions are related, but the scope of each query is markedly different. Question 1 is not a statistics question as defined by GAISE because it is not answered using data that vary; it is answered by comparing only two individual cookies. On the other hand, questions 2 and 3 are answered using multiple cookies of the same type, and the amount of cream in each will differ, even if only slightly. Moreover, questions 2 and 3 suggest generalization to the broader group of all regular and double-cream cookies. In addition, question 3

Table 1 Measures for 10 Regular and 10 Double-Cream Cookies

Group	Regular Cookies with Filling	Regular Cookies without Filling	Double-Cream Cookies with Filling	Double-Cream Cookies without Filling	Cream Ratio
Group 1	115	82	146	82	1.94
Group 2	116	83	146	82	1.94
Group 3	116	82	147	82	1.91
Group 4	114	85	146	81	2.24
Group 5	114	84	147	83	2.13

suggests comparing the mean (i.e., average) amount of cream, a population parameter, in the two cookie types. In whole-group conversations, students noted that questions 2 and 3 "ask the same sort of question" but that question 3 suggests "what to do" more clearly than question 2.

The comparison of student-authored research questions helped students—both those in methods courses and those in introductory statistics courses—recognize the importance of posing questions succinctly, in ways that are clearly measurable and testable. Further, the task emphasized that statistical studies require asking statistical questions—those that can be answered using data that vary.

Data Collection

The concerns of our data-group students focused on limitations associated with sample size. In addition, students identified problems concerning data variation, instrument precision, and random sampling. These concerns are summarized below.

Sample size and variation. The data-group students agreed that sample size in the original investigation was "too small," although they struggled to articulate what "too small" meant. Several groups recommended repeating the original investigation with 50 of each cookie rather than 10. This recommendation led to a series of interesting questions. For instance, why not 100? Why not 500? How many cookies are needed? Questions such as these sparked animated debates in each of our classes. Although some students agreed that 50 cookies would provide a more accurate overall measure of regular and double-cream varieties than 10, others understood that statistical questions need to be answered using data that vary (Franklin et al. 2007). By challenging students with the question "How many measurements were actually taken?" students came to the conclusion that finding the average weight of 10 or 50 (or even 5 million) cookies provides a single measure of weight and, as such, fails to reveal how weight in a sample tends to vary from one cookie to another. When we asked students, "Is taking one measurement enough?" they responded that the original investigation failed to provide information regarding variation of cream weight "from cookie to cookie" for each cookie type.

Instrument precision. A number of students in both classrooms expressed concerns regarding the precision of the scales used to measure cookies. Note that the scales in **figure 2** measure mass to the nearest gram. Hence, conflicting results may be due in part to instrument precision. For instance, one methods student asked her classmates to consider the measurements in the first row of **table 1**. She pointed out that measures of 146.4, 81.5, 114.5, and 82.4, when rounded to the nearest gram, yield a cream ratio identical to that provided in the table: 1.94. Yet the unrounded measures yield a ratio larger than 2 (i.e., 2.02). Students' observations such as these suggested the need for scales that measure to the nearest tenth or hundredth of a gram.

Random sampling. Last, our students noted that the cookies in the original investigation came from the same package. As such, the cookies were manufactured on the same day, at the same time, on the same machinery, at the same processing plant. As one preservice teacher commented, "What if the icing station at the double-cream factory was dispensing less cream than usual on that day? What if regional differences existed at different factories?" (Student B, personal communication, Nov. 21, 2013). By failing to randomly sample cookies, the original investigation failed to account for possible variation associated with the cookie manufacturing process.

PHASE 3: REFORMULATING THE STUDY

In the final phase, we asked students to reformulate the original study. In small teams, they developed their own research questions (GAISE component 1) and described their own procedures for collecting (GAISE component 2), analyzing (GAISE component 3), and interpreting (GAISE component 4) cookie data (see the **sidebar**). Phase 3 culminates with students sharing their analyses and findings in short presentations to classmates. Here we highlight issues critical to the success of this phase. Specifically, we describe how students addressed their concerns associated with data collection, sampling, precision, and variation. Last, we highlight a statistical investigation that was popular among introductory statistics students that aimed to compare regular and double-cream cookies more rigorously than the original cookie investigation.

Sampling Concerns

Students commonly believe that factors such as sample size are more important than representativeness, so at the beginning of phase 3 we emphasized the importance of collecting data from a sample that adequately represents the population. Through online research, students learned that this particular brand of regular and double-cream cookies is manufactured at two different factories in the United States. They also learned that strict controls

Variable Name	estimated mean	degrees of freedom	t	p	confidence interval percent	confidence interval of difference
original2x.no.outliers double.no.outliers	6.334379 6.514118	224.6721	-4.734654	3.885102e-06	95	-0.2545469 -0.104931

Fig. 4 Student results of a *t*-test comparing the mean of twice the mass of regular and double-cream data sets showed evidence that double-cream cookies have more than twice the cream of regular cookies.

are implemented to ensure uniformity at each factory through a Total Quality Management (TQM) process (Manley 2011; Lusas and Rooney 2001). On the basis of these findings, students concluded that it was reasonable to assume that cookies purchased from a few different stores at different times could adequately represent the population of all regular and double-cream cookies.

Precision and Variation Concerns

Students determined that a precision scale capable of measuring mass to the nearest hundredth of a gram should be used to collect mass data from individual cookies. Unlike the original investigation, which provided students with a single measure of mass, the revised approach should provide students with access to data that vary.

Data Collection Concerns

Because purchasing packages of cookies and precision scales for multiple groups is cost prohibitive and because significant time is required to collect individual cookie data, we elected to collect cookie data that our students could subsequently use in their follow-up work. We measured the mass of cream and biscuits for 180 regular and 180 double-cream cookies purchased from different stores in two different states. Our data collection was informed by concerns that students expressed in phase 2. We provided access to the data from an online spreadsheet available at http://bit.ly/cookie-data.

Student Investigation

After concerns regarding data collection, sampling, and precision were addressed, small groups of students formulated their own research questions in phase 3. One group's research question is provided in question 4:

4. Is the mean mass of cream in all double-cream cookies equal to twice the mean mass of cream in all regular cookies?

To address this question, the student group performed a hypothesis test with the provided data after deciding to identify and remove outliers. Using a two-sided (i.e., two-tailed) *t*-test, the group determined whether there was evidence that the mean of twice the mass of the regular cream is different from the mean double–cream mass. Using R, a freely available statistics analysis software (go to http://www.r-project.org), students analyzed data from http://bit.ly/cookie-data to generate the results provided in **figure 4**.

In their written work, students summarized their interpretations of these results, connecting the *t*-statistic back to regular and double-cream cookies. After a class discussion of these results, one team member wrote the following summary: "The *t*-test gives a *p*-value of 0.000004. This means that if we randomly collected 360 cookies many, many times, where the double-cream cookies are truly twice the regular cookies, only 0.0004% of those samples would produce the difference we observed in our data. Because this probability is so small and because the mean cream weight of the double-cream cookies in our sample was larger than twice the mean cream weight of our regular cookies, we can conclude that, on average, the

THE COOKIE CONUNDRUM: PHASE 3

Formulating a Question

1. What question does your group wish to answer with the revised cookie data? State the aim of your follow-up investigation as a one-sentence research question.

Collecting Data

2. What issues should you address when collecting data that will help you answer the question that you formulated?

Analyzing Data

3. Analyze the cookie data available at http://bit.ly/cookie-data, specifically addressing the research question 1. Describe the steps of your analysis in several complete sentences. For each step, provide a brief justification for its inclusion in your study.

Interpreting Results

4. What do the results of your analysis appear to indicate? Justify your conclusions using evidence from your data analysis. Include data displays (e.g., graphs, output from statistical software, etc.). In several complete paragraphs, give thoughtful interpretations of these displays and the other data analysis that you performed.

cream in double-cream cookies for this brand is actually *more than twice* the cream in the regular cookies" (Student C, personal communication, Nov. 21, 2013).

These findings were consistent with those of other student groups in both our classes. Without exception, students found reasonable evidence to conclude that double-cream cookies, in fact, had at least twice the cream of regular cookies. Arguably more important, students in both classrooms recognized that the media failed to tell the whole story.

NEXT STEPS AND IDEAS FOR FURTHER STUDY

Engaging Students at Various Levels

Connecting content from the mathematics classroom with news stories in the popular media provides students at any level with opportunities to use tools that are accessible to them to strengthen their understanding of statistics. Students in earlier grades can analyze cookie data using comparative displays (box plots and stem-and-leaf plots); measures of center (mean, median, and mode); measures of spread (variance, standard deviation, and interquartile range); and position (quartiles and percentiles). More advanced students can use resampling and bootstrapping methods or more traditional hypothesis testing such as that presented in the previous section.

Exploring Other Topics

The Cookie Conundrum project demonstrates ways in which new teaching and learning ideas can be generated in the space created by the overlap of popular media, teacher blogs, and instructional objectives. Certainly, such investigations are not limited to the study of cream-filled cookies. The three-phase model explored here may be used to investigate a myriad of other media claims in the mathematics classroom. Indeed, accounts in the popular media are rife with categories of examples worthy of exploration in school classrooms. Consider the following news stories and advertisements:

- A teen recently claimed that the U.S. government would save $400 million by changing fonts on government documents from Times New Roman to Garamond (Miller 2014). Is this true? What mathematical and statistical claims are made in the media story?
- A news report claims that students are not attending school dances because of their social media use (Pawlowski 2014). Is this claim warranted? What role did data, mathematics, and statistics play in this story?
- A popular sandwich chain was recently sued for selling footlong subs that were shorter than expected (Arumugam 2013). Was such a lawsuit warranted? What role might statistics play in the controversy?
- A popular brand of gum is marketed as producing "double" bubbles. Does the size of bubbles generated while chewing this brand of gum result in bubbles that are any larger than competing brands of bubble gum?

The Cookie Conundrum project provides powerful learning opportunities for students and their teachers. First and foremost, the project empowers students to question news reports that make use of mathematics and statistics. Because today's students live in an age of unparalleled access to data and technology, they need opportunities to grow as critical consumers of information. Classroom activities that encourage students to analyze data and statistics in media literacy contexts—news articles, blog posts, video analyses, statistical reports—address calls for instruction that is both "robust and relevant to the real world" (CCSSI 2010). As students complete the Cookie Conundrum project, they are surprised to uncover reporting that misrepresents facts. Moreover, they understand that journalists and readers need solid statistical understanding to report and respond to quantitative studies in an educated manner. Indeed, the project illustrates the need for quantitative literacy among all our citizenry.

Teachers who provide opportunities for students to make meaning of the complexities found in the four components of statistical studies (as outlined by the GAISE report) will help those students decipher warranted and unwarranted statistical claims—whether the claims are found in the news media, government reports, or the workplace. When we neglect to help students make meaning of these components, we unwittingly provide them with an artificial sense of the statistical design and investigation process. Through projects like the Cookie Conundrum, students are challenged to reason with and about data to reach well-reasoned conclusions in easily accessible and highly motivational contexts.

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MATHEMATICS teacher

Changing Classroom Instruction: One Teache Perspe

After many years of looking for effective teaching practices, a high school teacher now sees where the focus of change needs to be.

Penelope P. Tolle

"How easy is it to change how we teach?" There was a time when I asked that question every time I heard talk about what the ideal mathematics classroom should look like—engaged students, discourse, rigor, scaffolding, and questioning of various kinds embedded throughout each lesson. With the implementation of the Common Core State Standards, many of these behaviors now appear within the Standards for Mathematical Practice, on which students can be assessed and teachers will be held accountable. If changing the way we teach is an easy thing to do—and there are those who imply that it is easy—then why aren't all teachers equipped with the ability to attain all these desired practices in their classrooms when they first leave their university preparation programs and begin teaching?

The answer is that changing the way we teach is not easy, nor does it happen instantaneously. I am not the first to write about the difficulties of

DELTA; GEORGIOS KOLLIDAS/THINKSTOCK, CLASSROOM; JETTA PRODUCTIONS/THINKSTOCK

change, especially the time it takes to make changes (Edwards and Townsend 2012). I spent years making the changes in my teaching methods before realizing where change really needed to occur. So it is important that I first describe some of the major outward changes that I made—that many readers will probably see in their own teaching—before talking about the more effective change I made in my perspective on teaching.

EXPLAIN YOURSELF

I began changing my teaching methods in 2001, after eight years of teaching, when I was teaching developmental-level mathematics courses at the local community college. Having taught previously at an international private school, I knew just how important these early concepts would be to later courses. We were at the end of a unit on fractions, and, as I had been doing all semester, I encouraged students to share their different solution methods with the class.

In this case, the problem was subtraction of mixed fractions:

$$4\frac{3}{5}-1\frac{4}{5}$$

The last student to come to the board used a solution method that I had not anticipated. Subtracting the whole numbers gave him a 3, and subtracting the fractions gave him –1/5. But then he took the solution one step further than other classmates who had used the same method. He performed the signed number calculation, which resulted in the correct answer:

$$3+\left(\frac{-1}{5}\right)=2\frac{4}{5}.$$

At that point in my teaching career, I taught from the front of the room. I did little walking around and relied on students to let me know whether they had questions about what they were doing. Luckily, this student had not asked for help in solving this problem, because I might have encouraged him to use another more common solution method. But there was nothing wrong with his method. In his method, he had applied prior knowledge of signed numbers, which all the students in this class had already been exposed to.

I left that class knowing that I was not as prepared as I needed to be for each day's lesson and each student's possible contribution. I needed to stop teaching concepts in isolation and think more about where students had been and where they were going next. But what exactly does that change look like when I am preparing a lesson? The answer was just a guess on my part; I had no one and no model to reference. At that time, much was being written about using questions effectively, so that is what I focused on in changing in my lesson. I saw using questions as a way to hear and see more of my students' mathematical thinking.

Four years later, I was teaching algebra at a public high school. I was still teaching primarily from the front of the room, because I felt it was more effective. Answering a student's question from the front of the room allowed everyone to hear and benefit from what I considered to be a discussion. I could also call on someone else to answer a classmate's question, thus allowing a greater number of students to share their own ideas. My questions were improving, and I was using them also to guide students in connecting the different mathematical concepts and methods of solution. Typical questions were "What led you to use that method?" and "How is your thinking similar to [or different from] . . . ?"

But I stumbled every time a student offered

something incorrect to the conversation. I knew that it would be wrong to say, "No, that isn't correct." The solution suddenly came to me one day when I really wanted to know how a student had arrived at the wrong method and found myself wondering what that student was thinking at the time. After that, I began using the same cue for incorrect student contributions as for correct solutions: "Explain how you got that." I found that students, while explaining their own work, would often stumble on their own mistake and correct it; thus, they benefited from talking through their error.

CHANGES MADE

In summer 2007, I had the opportunity, as part of a lesson study immersion program, to go to Japan, whose students have done extremely well on international tests. I spent two weeks observing mathematics being taught at different levels, from the primary through the middle school grades, at schools throughout the country and witnessed both the classroom and postobservation segments of the lesson study process. While there, I also had the opportunity to observe and talk with mathematics education professors and preservice students at two universities. This trip gave me the teaching model that I had been looking for. But what I discovered on my return was that trying to put what I observed in Japan into practice in America was not an easy matter.

I continued to make changes to my own teaching, drawing on what I had observed and had been a part of in Japan but also on books and articles about lesson study by Catherine Lewis (2002) and others. Smith and Stein, in their *5 Practices for Orchestrating Productive Mathematics Discussions* (2011), provided an outline of a mathematical lesson that is extremely similar to the Japanese model. I did what I could to plan and implement lessons following a combination of all this input. I drastically reduced my time at the front of the room; added space on my seating chart so that I could circulate around the room and make notes about how students were approaching and solving problems; and tried not to assist them or direct them when they got stuck. I rewrote old problems to increase rigor, aligned problems better with objectives, and encouraged students to think on their own by choosing to hear what they had to say about their approach and then ask questions, rather than show them what to do. I organized the discussion at the end of a lesson to direct students to a deeper understanding of the mathematics. Professionally, I participated in workshops, took graduate-level courses, went to conferences, and participated in webinars. In retrospect, I notice that I had always been making my changes be all about changing me and what I was doing as a teacher.

SIX PRINCIPLES OF QUALITY MATHEMATICS INSTRUCTION

Then, as I was preparing to join the Common Core adventure, I read the article "Intellectual Engagement and Other Principles of Mathematics Instruction" (Peterson et al. 2013). After studying pre- and postlesson discussions between Japanese preservice teachers and their professors, the authors detailed six principles that seemed to them to characterize Japanese teachers' conception of high-quality mathematics instruction. The article helped me understand that what I really needed to do, and what I had seen done so successfully in Japan, required taking the focus off me, the teacher, and putting it where it needed to be—on student engagement with the lesson. I will use the six principles from that article to explain exactly what I mean by that.

What I really needed to do required taking the focus off me, the teacher, and putting it where it needed to be—on student engagement with the lesson.

The Intellectual Engagement Principle and the Goal Principle

Previously, my lesson plans had no greater goal than to keep students busy and sometimes even entertained with the mathematics skill or concept for that day. I had come to learn, however, that those kinds of activities are not the same as intellectually engaging students in mathematical thinking. A question that every teacher needs to ask when developing any lesson plan is, What is the mathematical goal of the lesson, and how does the activity encourage thinking that leads students to achieving that goal?

Good instructional packages need solid mathematical goals. For example, it is common in exponential function units for teachers to have students do a candy or penny activity, in which a number of candies or pennies are removed according to what is visible on the surface of the coin or candy. But what is the mathematical goal of that activity? Yes, it engages students, but is it intellectually engaging, and what mathematics should the students learn by the end of the activity? What

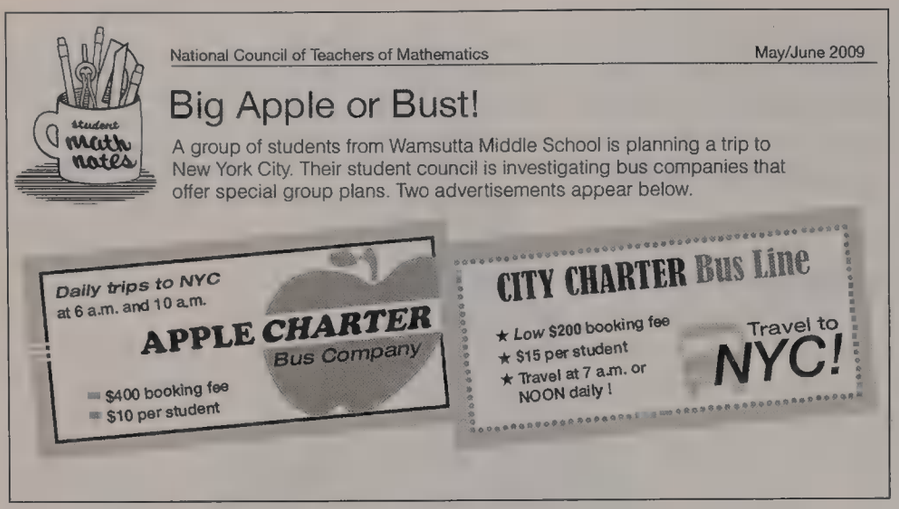
National Council of Teachers of Mathematics
May/June 2009
student math notes
Big Apple or Bust!
A group of students from Wamsutta Middle School is planning a trip to New York City. Their student council is investigating bus companies that offer special group plans. Two advertisements appear below.
Daily trips to NYC at 6 a.m. and 10 a.m.
APPLE CHARTER Bus Company
$400 booking fee
$10 per student
CITY CHARTER Bus Line
★ Low $200 booking fee
★ $15 per student
★ Travel at 7 a.m. or NOON daily !
Travel to NYC!

Fig. 1 A rich activity begins by engaging the students.

mathematics should we hear students discussing and see them doing as we watch and listen to them? Are they intellectually and mathematically engaged in the activity?

A few years ago, I removed the M&M's® problem from my lessons and implemented a different focus on the teaching of the exponential function, a direction that turned out to be more in line with the Common Core Standards. I have connected the teaching of the exponential function to the linear function, thus allowing students to discuss compare, and analyze the similarities and differences. Having this connection allows the exponential function lessons to flow nicely from what students already know about the linear function.

The Flow Principle

When I was looking for lessons that would engage students with rich mathematical content (I was not yet comfortable with creating these myself, nor did I have the time), Edward Nolan, then one of the editors of the *Student Math Notes* (now called *Student Explorations in Mathematics*) sent me a handful of activities that he thought might be what I was looking for.

One was a well-structured activity called Big Apple or Bust (Dixon 2009). Although a bit wordy, it had everything needed to best demonstrate the flow I had seen in Japanese lessons, and it had the goal I wanted for an introductory lesson for systems of equations. It had an engaging beginning (see **fig. 1**), connected a number of different representations, and offered a way to pull everything together in the end. At the same time, it got students thinking about the need for an algebraic equation to determine the solution to a system of equations. Most important, it started at an easy level of understanding and led students down paths of deeper mathematical thinking.

The Unit Principle and the Adaptive Instruction Principle

For all students to enter a problem or activity easily, it must have an obvious connection to topics learned in previous units and a simple-enough start that even the weakest students feel comfortable trying something.

The Preparation Principle

The preparation of a complete lesson (what I also call an instructional package) is the last of the six principles. In the Big Apple or Bust activity, students are asked to decide on transportation for a trip to New York City, and they are presented with two ads from two different bus companies, each one offering a flat rate and a price per student. The activity includes a table, on which the independent variable (number of students) increases by fives. The x-axis scale on the graph, also included, increases by tens. Because of the scales chosen in the table and graph, students are not able to state the actual point of intersection—the aspect of this activity that I found appealing. To answer the question "For what number of students would the cost of the two bus companies be the same?" students are forced to think about the meaning of the values on both sides of where the intersection value really is, both in the table and on the graph. It had never occurred to me to set the importance of the idea of finding the intersection value in the first lesson of the unit by making that intersection point not easy for them to determine. Finding the intersection required some mathematical thinking on their part, but nothing that was beyond their ability.

Because the actual intersection point is not known right away, we are able to explore other aspects of the systems topic as well. For example, I asked students to explain how they know when the cost of one company is more than the other company. The activity is very rich mathematically and did not need the large number of questions that were included in the original activity. In the activity sheet I gave out, I asked students to write an equation to represent the cost of each bus for n number of students. I wanted to know whether they could use their equations to figure out for how many students the cost of the bus companies would be the same. The rest of the original questions did not appear on the activity sheet but instead became part of my whole-class discussion.

I have used this activity for a couple of years, and the discussion generated by this activity is always rich. Students come up with many ideas about how they can find the exact number of travelers that would make the cost of the bus trips the same and often use one representation (a table or a graph) to predict the value and another

representation to check or justify their guess (the two equations). They offer sound mathematical arguments for which bus company would be better if a large, or small, number of students travelers were going on the trip. A whole-class conversation also revealed that the wording of my question—"Which would be the most economical?"—is not always going to give the same or most obvious answer for all students. For some students, a cheaper cost is not necessarily the most desirable company to go with, and that is a bias that developers of mathematics tests would need to be aware of.

A CHANGE IN FOCUS

Lessons do not have to be fancy. In a recent unit, students learned to graph a piecewise function by first drawing the entire graph and then erasing the portion of the graph that was not in the domain. When I was confident that they understood the importance of the domain aspect of the function, I reversed the process; I gave them a graph of a piecewise function and asked them to write the equations and corresponding domains for each section of the function. I knew that they could do it, but they would have to think backward, and some did struggle with that. But my students now sit in groups of three or four, which I rearrange from time to time to make sure that they are working effectively, while I move among them to hear their conversations and observe their work.

Change does not happen easily or quickly. Now my focus is on what my students are doing and thinking mathematically.

Change does not happen easily or quickly. I have made important changes, and now my focus is on what my students are doing and thinking mathematically. The focus is no longer on me and the mathematics that I think that I have to explain to students. Rather, it is on the students and the mathematics that they learn in the problems I select for them to work through. My lessons have a single mathematical goal that is connected to previous mathematics concepts, thus providing students a place to enter the problem comfortably. And I now look forward to seeing scores on the large-scale tests that my students will need to take.

Becoming a more effective teacher has been a long journey but a very worthwhile one. Mathematics education in the United States is currently in a time of transition and change. It is a perfect time for changing more than just our mathematics standards.

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USING REPETITION to Make Ideas Stick

Repetition of a key phrase that encapsulates a big idea can be a valuable teaching tool.

Alicia N. Lykins

"Natural selection is the vehicle for evolution." My high school biology teacher repeated this phrase over and over and over again during our unit on evolution. How is it possible that twenty years later I can remember that phrase verbatim but not the teacher's name? Over the years I have thought about this on several occasions. That sentence stuck with me, and the more I have continued to learn about the concepts of natural selection and evolution, the more sense that statement (or the intent of that statement) makes to me. Yet it took ten years for me to realize that if the strategy worked on me, maybe it would work in my own classroom.

I now teach sixth-grade mathematics. My students are younger and obviously are learning an entirely different concept than I was, but I wondered, If I use that same strategy of repeating the same "big idea" in the same way over and over and over again, is it possible that it just might stick? Is it possible that my students too would carry that phrase with them, go back to it, and connect new knowledge to it to create an even better understanding of the big idea? I remember that repetition as being incredibly annoying as a student in that biology class, but if it works, I could gladly be annoying as a teacher!

APPLYING THE REPETITIVE PHRASE STRATEGY TO MATHEMATICS CONCEPTS

Probability

The first time I used this strategy was during a unit on theoretical versus experimental probability. I knew from previously released versions of the sixth-grade math Ohio Achievement Assessment (OAA) that there would likely be a question that required students to understand that their results, the experimental probability, would approach the theoretical probability the more they repeated an experiment.

An example of one of these types of questions from the spring 2010 Mathematics OAA (Ohio Department of Education, p. 36) test was this: "Derek and Meredith are playing a game with the spinner shown. Which activity would best show that the spinner is fair? [A spinner divided into four equal sections was shown]." Students had four choices. Choice A was "Spin it one time and see which section it lands on." Choice B was "Spin it four times and see if it lands on each section once." Choice C was "Spin it 60 times and see if it lands on 'Go Again' about 30 times." Choice D was "Spin it 100 times and see if it lands on each section about 25 times." The concept that students need to understand is how a valid experiment design can yield results close to the theoretical probability. They need to distinguish the validity between spinning four times and spinning 100 times in finding the results they expect.

I taught the concept as I typically would—with hands-on activities, games, notes, and definitions with examples—and gave a quiz. To my surprise, about half the class missed the question. I knew it was time to drive this idea home.

After the quiz, I had the students take some more notes. This time I had them write down the phrase, "The more trials in your experiment, the more accurate your experimental probability will be." I said the phrase as they wrote it. I said the phrase again. I said the phrase again. I had them underline words such as *more* and *accurate*. I had them repeat the phrase after me. I said the phrase again. Pretty soon I had their attention because they were all wondering why I sounded like a broken record. I said it in different voices, I sang it, I said it leaving out words and having them fill them in. We looked at sample questions and repeated the phrase to help us find an answer that seemed to fit what we were looking for. The most important thing was not changing the phrase. I did not say it differently; I did not rearrange the words or use different words. I repeated it the exact same way every time. Every time we had a probability question, we repeated the phrase. I hoped it stuck, and it appeared to have worked when I tested the students; almost every one got the question correct. I had gotten the concept to stick—at least for a week or so!

Fast forward six months to last-minute test preparation for the OAA, which was given in May. I gave my students a study guide that included the same phrase, but I left out some of the words and put a blank for them to fill in. Sure enough, they all remembered it. We talked about the concept again, and they even remembered important related vocabulary words such as *theoretical* and *experimental*. They were able not only to choose the correct answers from multiple choice questions but also to express their mastery of the concept by answering an extended response question in which they had to describe how they would design a probability experiment. They all used large numbers of trials in their experiment designs. Now it is my hope that as they go through their future mathematics courses, they will have that "big idea" to go back to and that, as they progress, they will keep connecting new knowledge to that boring little phrase they learned in sixth grade.

Inverse Operations

I also used the repetitive phrase idea when teaching inverse operations. From previous courses, all my students were familiar with fact families—that is, groups of three numbers that are related by addition and subtraction or multiplication and division. I asked them to recall how fact families worked ("How are the numbers 63, 9, and 7 related? Can you multiply two of them to get the third? Can you divide two of them to get the third?") So I used this phrase: "When trying to find a missing factor in a multiplication problem, you can divide." Every time we came across a problem where we needed to find one of the factors of a multiplication problem in which we knew the product, I would ask the question, "If we need to know what two numbers we can multiply together to get a certain product

I taught the concept as I typically would—with hands-on activities, games, notes, and definitions with examples—and gave a quiz. To my surprise, about half the class missed the question. I knew it was time to drive this idea home.

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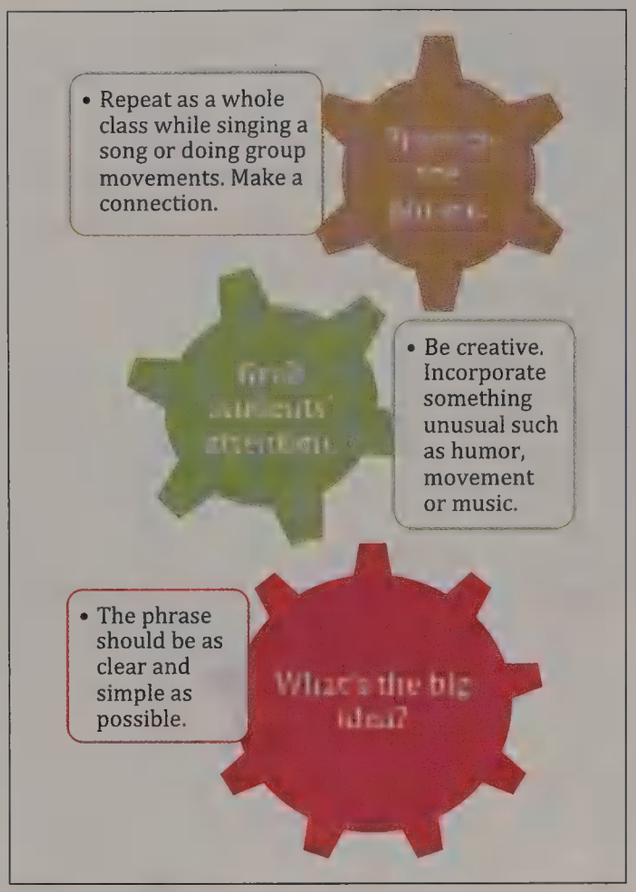
• Repeat as a whole class while singing a song or doing group movements. Make a connection.
• Be creative. Incorporate something unusual such as humor, movement or music.
• The phrase should be as clear and simple as possible.

Fig. 1 Keep the concept as simple as possible and focus on the big idea.

what can we do?" By the end of the year, every student knew the answer: "Divide!" It was not much of a stretch, then, to extend that idea to division problems in which we were missing a divisor. Again, the key was using a consistent phrase. This was extremely helpful when it came to one-step equation solving and can then be extended to two-step equation solving.

In elementary school, the use of repetitive songs to help children remember concepts is commonplace and is usually very effective. Unfortunately for many students, this strategy is generally not used in later grades. A group of mathematics teachers at Westerville South High School in Westerville, Ohio, have taken this approach to a new creative level. These teachers have created a group of math rap videos on YouTube™, set to current rap music hits. The raps teach concepts such as graphing linear equations, solving systems of equations, rules of exponents, solving quadratic equations, trigonometry, and factoring quadratic equations. They are extremely catchy and have a chorus that repeats. The repetitive nature of the lyrics, in addition to the fact that students can listen to the raps over and over again while working on homework or needing some help, then becomes a way to help remember basic concepts that can be built upon as understanding grows.

Several key components ensure that this repetition would result in a long-term memory that could be retrieved.

RESEARCH UNDERLIES THE STRATEGY

So why does this strategy work so well? I began to look for research that would help me refine this idea of repetition and make sure that it was sound practice, not just something that would get students through the test. As one would expect, vast research has been conducted regarding repetition and its effect on memory and learning. Further, the research did acknowledge that repetition is truly a valuable teaching strategy. But there were several key components that would ensure that this repetition would result in not just a short-term memory but also a long-term memory that could be retrieved; new information could then be "hooked" to it.

Focus

First, teachers have to continually focus students' attention on important information to be learned and to engage them in actively rehearsing that information until the students securely retain it in long-term memory (Banikowski 1999). So when deciding which phrase I want to use for repetition, I need to be sure that I am focusing solely on the "big idea" that I hope will stick with students. The concept should be as simple as possible (see **fig. 1**).

Creativity

Once I have the big idea and the repetitive phrase, I need to think of creative ways to introduce it and have the students rehearse it and make connections with it. Richards (2008) writes, "Information 'grabbed,' or made meaningful, moves on to short-term memory. Our brains are programmed to pay attention to the unusual—something different. Incorporating novelty such as humor, movement, or music into strategies helps the information attract our attention."

This year I also tried to make use of music, specifically rap as at Westerville, to teach mathematics concepts. A great resource that I used extensively is Flocabulary (www.flocabulary.com), which offers highly repetitive raps that teach a variety of concepts and formulas. Grabbing a key line from the chorus of the rap song to use as my repetitive phrase works because most students this age memorize songs easily on their own. I would play the song once and have them simply listen. Then I would play the song again two more times and have them fill in missing lyrics as they listened. The lyrics are also usually on the screen so that students can see them as well as hear them and write them. I would play the song again the next day or as often as we were still working on the concept. Days later I would hear students

singing the chorus from songs about concepts such as finding the area of circles.

Practice

The next phase is practice. We all know that students must practice first with assistance and then on their own. The key here, again, is using that same phrase. According to Brabeck and Jeffrey (2013), "Researchers who have investigated expert and novice performance have uncovered important distinctions between deliberate practice and other activities, such as work, play and rote repetition. . . . Effective practice is deliberate. It involves attention, rehearsal, and repetition and leads to new knowledge or skills that can later be developed into more complex knowledge and skills."

During the practice phase, teachers should take any opportunity to develop a positive emotional experience for students. Wesson (2012) noted some key factors for memories. First, the stronger the emotions connected to an experience, the stronger the subsequent memory. In addition, learning experiences become more memorable when social-emotional memories are part of the learning event, which is why cooperative learning is such a powerful memory enhancer in schools.

CLASSROOM APPROACHES

Several approaches can be used when trying to get a specific phrase about a "big idea" into students' long-term memory. When the information or big idea phrase is first introduced, it is important to call it to the attention of students. For example, when I am lecturing or giving notes, I really do not have the attention of most of the students in the room. They may be copying what they see on the board or writing what they hear me saying, but they are not actually paying attention. It can be easy to keep talking as students are writing, assuming that they are listening at the same time. But as teachers we know that is not the case.

When I repeat the same phrase over and over, long after students have written it down, students begin to realize that I am saying something important. Banikowski (1999) writes, "Many theorists believe that attention plays a key role in moving information from the sensory register to working (short-term) memory. Basically, whatever an individual pays attention to moves into working memory [Ormrod 1998]. Anything in the sensory register that does not get a person's attention disappears from the memory system" (p. 2). I will have students repeat the phrase after me.

Banikowski also suggests using strategies such as Think-Pair-Share for students to rehearse the information just presented. In this approach, students first think about the idea on their own. Perhaps they could keep a journal about it or write down some questions or some new understandings. Students then pair up with another student or a small group of students and share their thoughts with one another. These group or pair discussions can then be shared with the larger group.

THE GOAL: LONG-TERM AS WELL AS SHORT-TERM RETENTION

Repetition has received a bad rap in today's academic world. Multiplication facts are a classic example of the "drill and kill" repetition method of teaching that may or may not be considered by some to be beneficial in some ways. However, there seems to be sound evidence that repetition does in fact play a vital role in memory. The key seems to be that the information that is being repeated is as focused and simple as possible so that students are not overloaded with unnecessary information. Once the big idea or concept can be reduced to a simple phrase that students can remember, they must make an emotional connection with the information so that it becomes something that will be transferred to long-term memory and then later retrieved when they learn new information that can be connected.

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Anne Quinn

Using Apps to Visualize Graph Theory

Once my teaching colleagues and I discovered how much algebra can be done by using cell phone apps, we had to constantly remind students to put their cell phones away. However, during a recent graph theory unit, when our laptops were not working, I found myself asking students to take their phones out again to follow along in class. The apps discussed in this article can be used on any iOS device, including iPads®, iPods®, and iPhones®.

INTRODUCTION TO GRAPH THEORY

Vertex-edge graphs are useful in modeling real world learning and so can be an important part of the high school curriculum (Hart 2008). Holliday (1991, p. 95) states: "Graph theory . . . [lets students] experience and understand many different proof techniques for important real-world applications." Although not as popular as algebra, vertex-edge graphs continue to deserve a role in the curriculum because they are necessary for solving many problems of the future, including project management, network flow, and transportation networks (Hart 2008; Cornell 2014). Graphs can be a unit incorporated into a geometry course or an integrated course. An Illuminations article (Bush 2014) suggests that explorations of vertex-edge graphs fulfill Common Core State Standards for Mathematics (CCSSI 2010), such as some of the Standards for Mathematical Practice—"Use appropriate tools strategically" (SMP 5, p. 7) and "Look for and make use of structure" (SMP 7, p. 8). In addition, the basic ideas are accessible and fun for students of all ages.

For years, The Geometer's Sketchpad® (GSP) has helped students visualize both traditional geometry and graph theory (Quinn 1997). However, graph theory apps and the prevalence of iPads in class (Perry, Thrasher, and Lee 2014) offer an opportunity to supplement the classroom with inexpensive technology. Although this article emphasizes a free iOS app named Graphing (Altece 2012), any similar app could be used.

In graph theory, graphs are built from vertices and edges. Unlike algebraic graphs, vertex-edge graphs can be manipulated without changing the relationships. Isomorphic graphs are a cornerstone of graph theory. In these graphs, vertices adjacent in the first graph remain adjacent in the new one, as in **figure 1**.

Vertex-edge graphs have many practical applications, representing contexts such as routes to destinations, where the vertices represent cities or stores and the edges represent airline flights or roads.

Inexpensive graph theory apps help teachers present more professional lectures and activities. They also help students visualize, experiment, share answers, and develop insight for proofs. In particular, apps can be used for matrix representations, isomorphisms, bipartite graphs, and planar graphs. A bonus is that students are excited to use the Graphing app on these problems.

A basic skill in graph theory is to link matrix and graphical representations of graphs. For example, the matrix representation of the triangle in **figure 1** is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

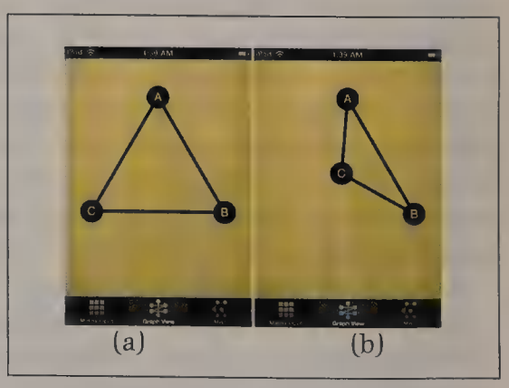

Fig. 1 These two graphs are isomorphic.

Edited by James Kett

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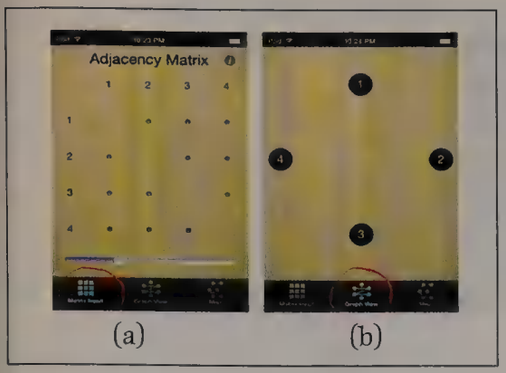

Fig. 2 The app has both a matrix and graphical mode.

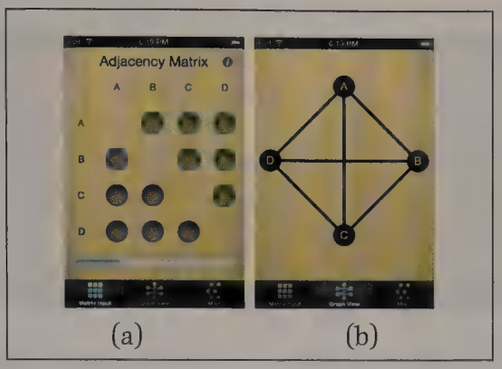

Fig. 3 Complete graphs include all possible edges.

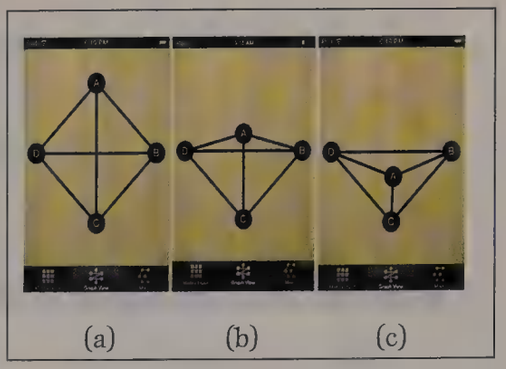

Fig. 4 Many isomorphisms can be created.

with the 1s signifying edges between A and B, B and C, and C and A and the 0s showing no edge from any vertex to itself.

Instead of allowing students to sketch freehand, a dynamic app such as the Graphing app requires students to create the correct matrix to draw the graph. To navigate the app, notice that the bottom bar lets users design in **Matrix Input** mode (see **fig. 2a**), view the graph in **Graphic View** (see **fig. 2b**), or switch from letters to numbers in **Misc**.

The interactive nature of this app lets students choose the vertices (up to twelve numbers or letters) and the edges and helps them visualize transitions between the representations. Consider the complete graph in **figure 3**, in which each pair of vertices is connected with an edge. If vertices represent people and edges represent friendships, this graph shows that each person has three friends.

Isomorphic Graphs

The Graphing app can create *isomorphic graphs*, which preserve relationships, by moving vertices. The app illustrates that graphs can be drawn differently without changing the relationships (see **fig. 4**).

The cube is another common example in both graph theory and geometry. When students model these edges on the Graphing app, they get the matrix and graph shown in **figure 5**. The image in **figure 5b** does not look like a cube. However, experimentation eventually leads to the isomorphic graph of a cube in **figure 5g**.

Another interesting example shows that a star is isomorphic to a pentagon (see **fig. 6**).

Once students use the app to transform graphs such as these, they have an easier time doing similar problems without technology—an important goal.

Bipartite Graphs

A standard topic in graph theory that also has many real-world applications is *bipartite graphs* (Cornell 2014; Greenwell and Seabold 2014). The vertices of a bipartite graph can be divided into two sets of vertices so that edges exist only between the two sets and never within the sets. Consider a

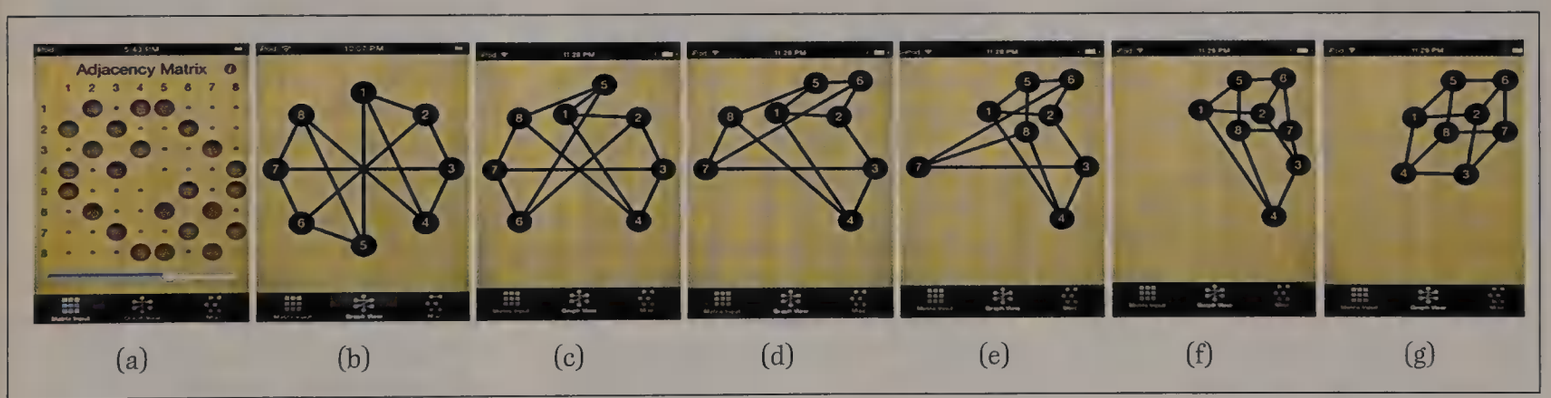

Fig. 5 Students can construct a cube.

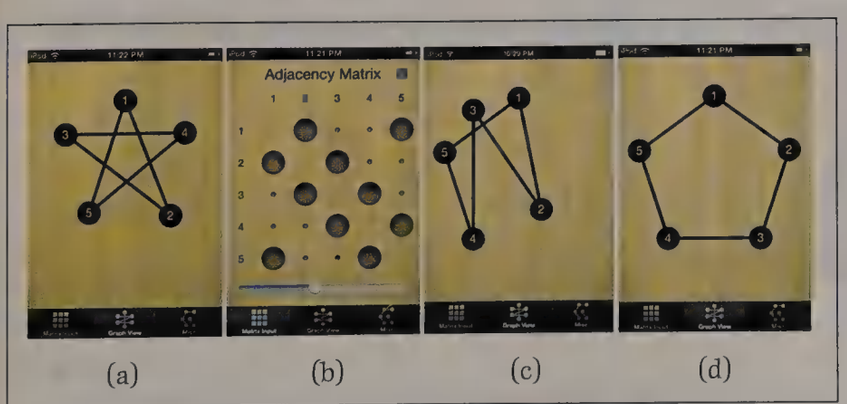

Fig. 6 A star can be transformed into a pentagon.

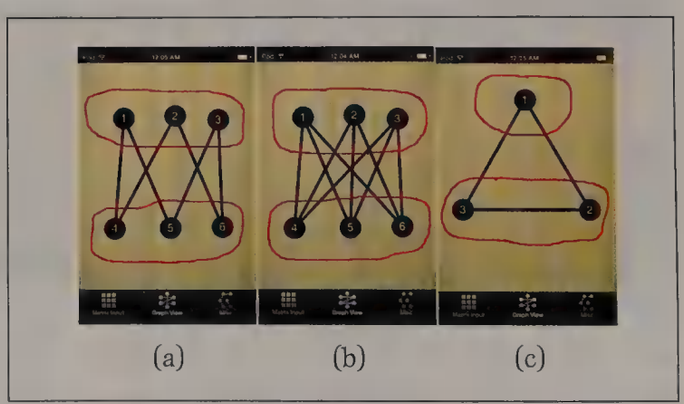

Fig. 7 Which graphs are bipartite?

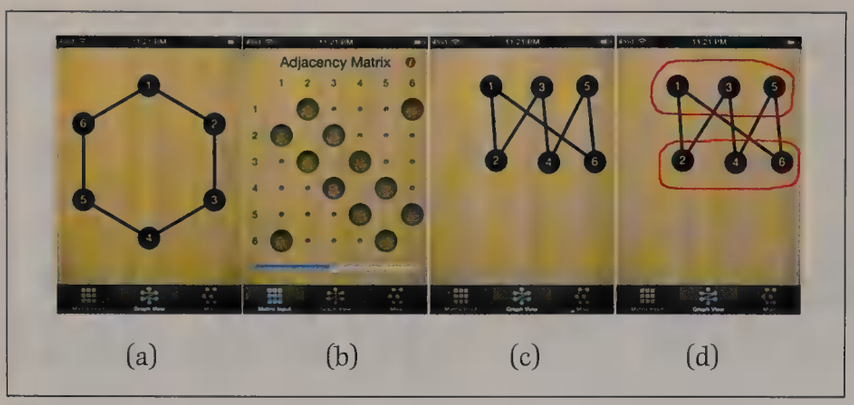

Fig. 8 C_6 is bipartite (**d**).

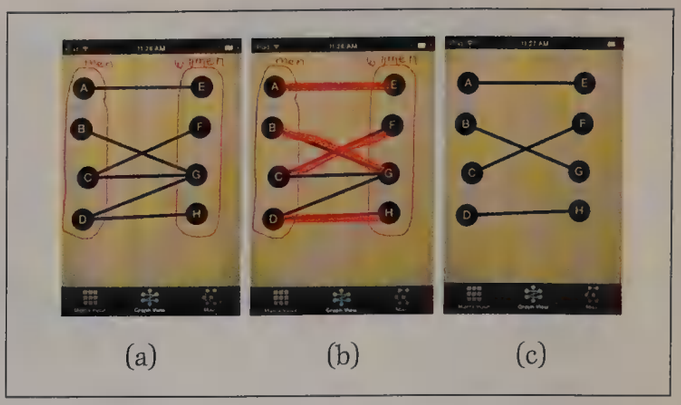

Fig. 9 The app can illustrate dancing partners.

few examples. **Figure 7a** is a bipartite graph. **Figure 7b** is a complete bipartite graphs, in which all possible edges are drawn between the two sets. **Figure 7c** is not bipartite.

Although it would be impossible to make the pentagon C_5 (see **fig. 6d**) into a bipartite graph, students can show that the hexagon cycle C_6 (see **fig. 8a**) is bipartite by identifying the vertex sets {1, 3, 5} and {2, 4, 6} (see **fig. 8d**). Actively constructing the adjacency matrix gives students insight into which elements should be placed into each of the two sets required for bipartite graphs.

Bipartite graphs have many real-world applications (Greenwell and Seabold 2014; Chartrand 2012; Hart 2008), such as this matching problem:

> There are 4 men (Andrew, Bill, Caleb, and David) and 4 women (Elizabeth, Frances, Gen, and Helen) at a party. The potential dancing partnerships are shown in the bipartite graph (see **fig. 9**). **Figure 9b** shows one pairing solution: (*A*, *E*), (*B*, *G*), (*C*, *F*), (*D*, *H*).

Many other applications exist, including matching jobs with qualified applicants, pairing students with compatible lab partners, or even illustrating that sometimes no perfect pairing exists.

Although units on *planar graphs* (graphs that can be redrawn so that no lines cross) usually focus entirely on pure mathematics, applications do exist. For example, in printed electronic circuits, if the vertices are terminals and the edges are connections, we would want to avoid paths crossing to avoid short circuits (Chartrand 2012). Determining planarity is also helpful for some scheduling problems. However, planar graphs are one of those pure math topics that students enjoy simply for the challenge, as evidenced by the many addictive planar graph apps that have been created (see Vivek [2012]; GetApps [2013]; and Rakesh [2013]).

A classic problem is to prove whether or not K_5 (see **fig. 10**) is planar. The Graphing app allows students insight into a proof by contradiction by showing that one crossing remains. **Figure 11** shows a similar problem (Rosen 2012, p. 725, no. 4). However, with one missing edge, this graph is planar.

Students can use an app to show that the bipartite graph in **figure 12a** (Rosen 2012, p. 738, no. 3) is also planar. First find the matrix representation (see **fig. 12b**) and then manipulate it (see **fig. 12c**). Interestingly the graph is also isomorphic to the cube (see **fig. 12d**).

Figure 13a is another example of a graph that is planar (see **fig. 13c**). It can be helpful to ask students to hypothesize about which vertices should be moved before testing their conjectures.

Application Problem

To determine when to offer classes so that students do not have time conflicts, use the graph in **figure 14a**, in which vertices represent classes (Chartrand 2012, problem set 9.1). Edges are constructed between classes that will have students in common and consequently should be in different time periods. Since the graph is planar (see **fig. 14b**), the four color theorem guarantees that we will need four or fewer time slots. (Possible solution: assign class 1 to 8 a.m.; class 2 to 9 a.m; classes 3, 5, and 7 to 10 a.m.; and classes 4 and 6 to 11 a.m.)

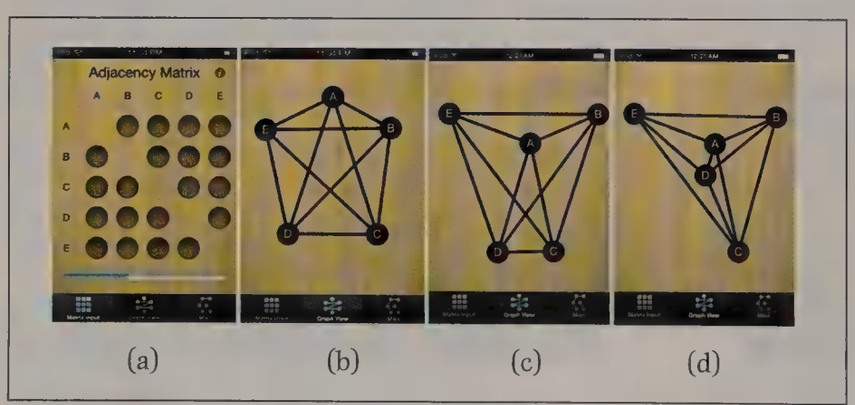

Fig. 10 The app suggests why K_5 is not planar.

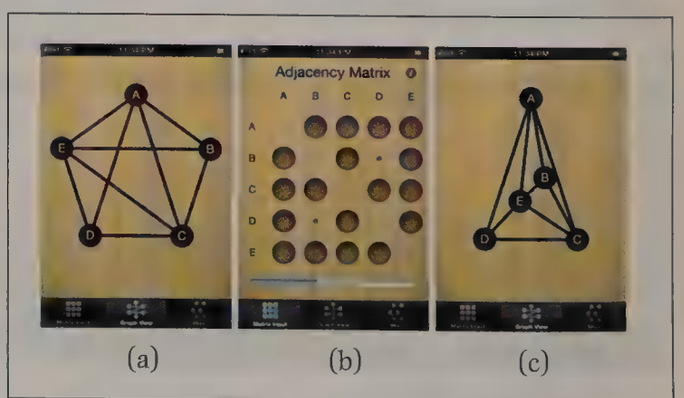

Fig. 11 Removing one edge from the graph in **figure 10b** makes this graph planar.

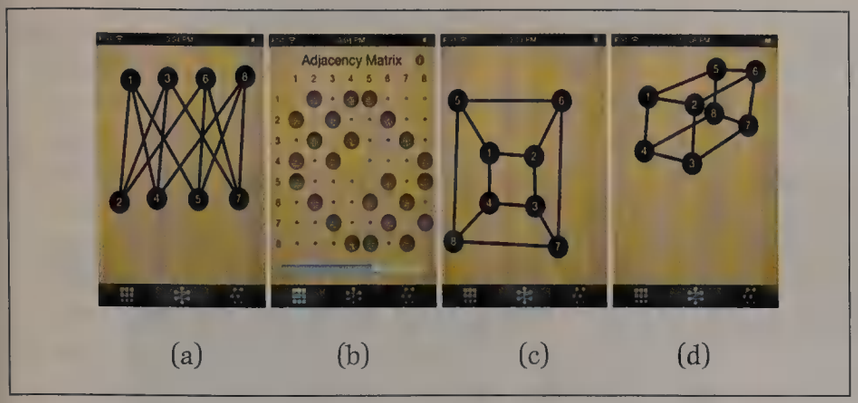

Fig. 12 This bipartite graph is both planar and isomorphic to the cube.

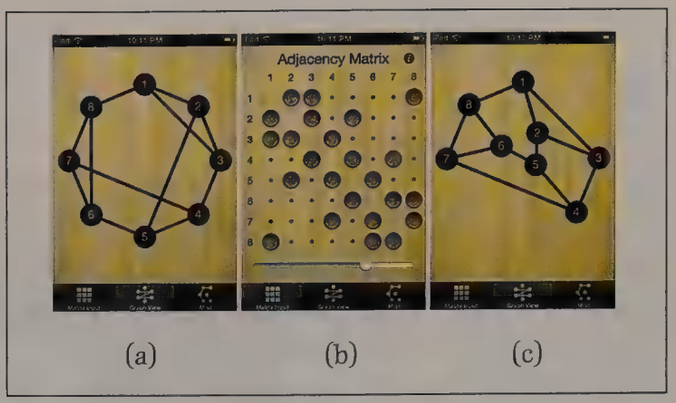

Fig. 13 Is the graph planar?

TIPS FOR COLLECTING STUDENTS' WORK

To print their work, students can create a cloud drive such as Dropbox (see **fig. 15a**). To open a new Dropbox account, first go to www.dropbox.com and follow the directions. Then download the Dropbox app from the AppStore on the iPod or iPad. Open up the Dropbox app, choose the **Settings** wheel icon (circled in red on the bottom panel in **figure 15b**), and then link the device to the Dropbox by tapping the button that says **link** (or **unlink** if already linked).

To synchronize the mobile device and the Dropbox account in the cloud, log into Dropbox on the mobile device and fill in the username and password. To take screen shots of students' work on the mobile device, first set up the device to receive automatic uploads of all pictures that it takes (see **fig. 16**).

Next, to take the screen shots of students' work on the apps, hold the power key and the **home** button (the circular button at the bottom of device) at the same time. If students have followed all the previous steps, screen shots will appear and load automatically (in **Photos** mode in **fig. 17**) from the iPod or iPad to the computer Dropbox account.

Finally, log in to the Dropbox account at www.dropbox.com. Choose the **Photos** tab, and all photos will appear. If they do not upload immediately, try reopening the app on the handheld device or switching between the **Files** and **Photos** icons on the computer (in **fig. 18**). To print the photos, double-click on a photo and then right-click to download. Next, open the file in another available program (such as **Paint**), highlight it if you wish, save it as a jpg file, insert it into a document, and print. Additional directions on how to share with Dropbox are included in Perry, Thrasher, and Lee (2014).

APPS FOR GRAPHING AND SHARING

Discrete mathematics remains an important content area in high school (Hart 2008), especially for students intending to pursue STEM majors in college. Although graph theory can be used to solve complicated modern problems (such as transportation, scheduling, network flow, distribution systems, and network survivability), the basic ideas are accessible and fun for students of all ages.

The Graphing app helped students become excited about the content, investigate the interplay between matrix and graphical representations, make and test conjectures, and explore transformations that led to isomorphic

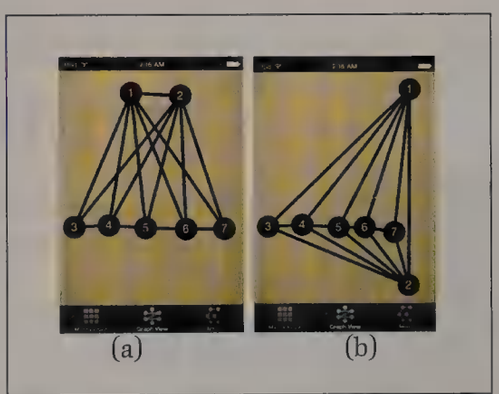

Fig. 14 The scheduling graph is planar.

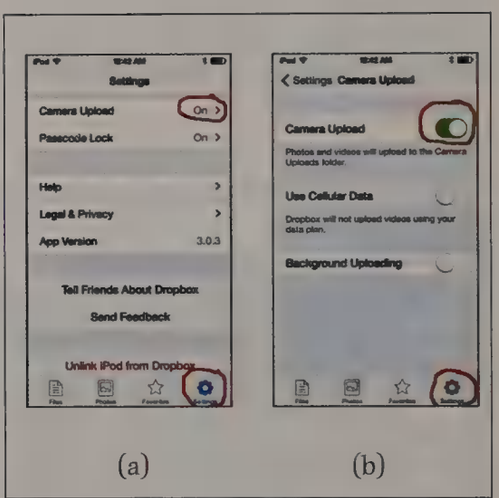

Fig. 16 Both iPod (**a**) and iPad (**b**) can upload pictures to Dropbox automatically.

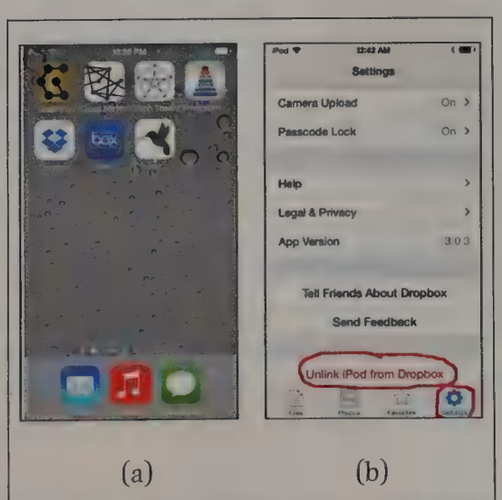

Fig. 15 Dropbox can be linked to mobile devices.

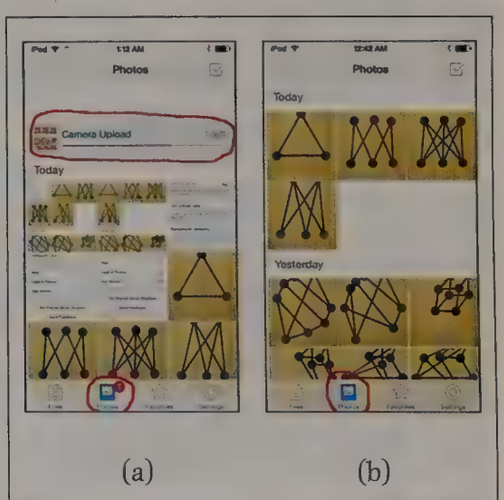

Fig. 17 Screen shots automatically upload to Dropbox.

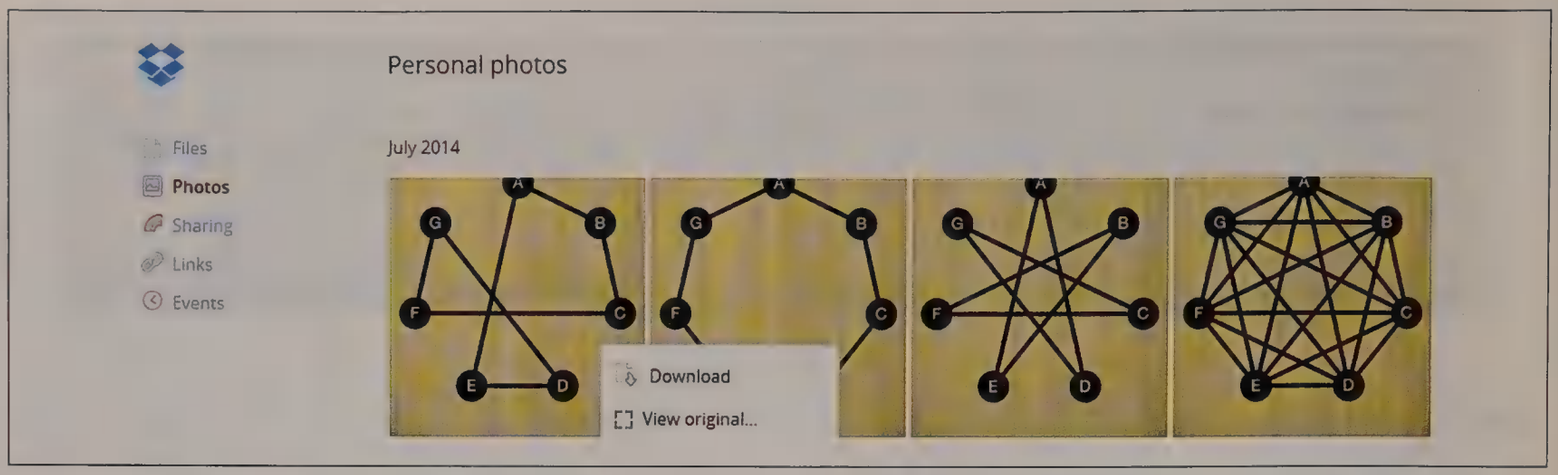

Fig. 18 Dropbox photographs can be printed.

graphs that were bipartite or planar. Apps such as Dropbox can be used for both teachers and students to document and share their results. In addition, the technology did not do the work for the students, who had to input matrices themselves. Instead, it helped them visualize and experiment.

A collection of nine additional sample problems along with several references is available at **www.nct.org/mt073**. Teachers may find these useful supplements, allowing their students to explore problems with technology, begin writing relevant proofs, and tackle contemporary applied problems in graph theory.

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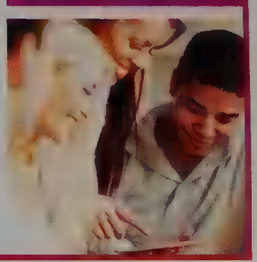

DELVING deeper

José N. Contreras

Discovering, Applying, and Extending Ceva's Theorem

Among the most commonly known points of concurrency are the four points of intersection classically associated with a triangle: the centroid, the orthocenter, the incenter, and the circumcenter. Armed with two extremely powerful theorems from elementary geometry—Ceva's theorem and its converse, here referred to simply as Ceva's theorem—I take my college geometry class on a journey to simplify and unify the proofs of classical concurrency theorems.

Giovanni Ceva was an Italian mathematician and hydraulic engineer who, using an argument based on mechanics, first published a proof of the theorem that now bears his name in a 1678 work *De lineis rectis* (O'Connor and Robertson 2012). Ceva was not the first to prove this theorem, however; an eleventh-century mathematician, Yasuf Al-Mutaman ibn Hud, first posed and proved the theorem (Holme 2010).

DISCOVERING AND PROVING CEVA'S THEOREM

A *cevian* is a segment, ray, or line that contains exactly one vertex of a triangle and a point, called a *division point*, of the opposite side or its extension. To help my geometry students—prospective secondary school mathematics teachers—experience the delight of discovering mathematical ideas, I pose the problem:

> Construct three arbitrary cevians of $\triangle ABC$—$\overline{AL}$, $\overline{BM}$, and $\overline{CN}$ (see **fig. 1**)—and compute $(AN/NB)(BL/LC)(CM/MA)$.
> Under what conditions would $(AN/NB)(BL/LC)(CM/MA) = 1$?

I initially restrict students' investigations to cevians whose division points lie on the triangle. The crossbar theorem and its converse assert that for any $\angle ABC$ a point P is on the interior of the angle if and only if $\overline{BP}$ intersects the interior of $\overline{AC}$. Consequently, in this investigation, any two cevians containing different vertices of a triangle must intersect in the interior of the triangle. Later extensions to the more general version of Ceva's theorem included cevians with division points on the extended sides and outside the triangle. This article describes only the restricted investigation.

My students eagerly use The Geometer's Sketchpad® (GSP) to construct a configuration like that in **figure 1** and investigate the problem. They drag any of the movable division points (N, L, or M) of the sides of the triangle until the product $(AN/NB) \cdot (BL/LC)(CM/MA)$ is very close to 1 (see **fig. 2**). This investigation leads students to conjecture that the product equals 1 whenever the three cevians are concurrent. Some students then drag vertices of the triangle to change its shape and size and repeat the process to confirm their conjecture. Others construct the configuration such that the three cevians $\overline{AL}$, $\overline{BM}$, and $\overline{CN}$ are concurrent and then compute the product (see **fig. 2**). Satisfied with confirming their conjecture and discussing in their groups how to word their proposition, they summarize their findings as follows:

> Three cevians $\overline{AL}$, $\overline{BM}$, and $\overline{CN}$ of $\triangle ABC$ are concurrent if and only if $(AN/NB)(BL/LC) \cdot (CM/MA)$ equals 1.

This astonishing relationship is Ceva's theorem.

As students notice, Ceva's theorem is a biconditional statement and therefore requires two proofs.

Edited by Maurice Burke

Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership; it appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net.

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I first have my students prove that if $\overline{AL}$, $\overline{BM}$, and $\overline{CN}$ are three concurrent cevians of $\triangle ABC$, then $(AN/NB)(BL/LC)(CM/MA) = 1$. Because this conclusion involves ratios, students intuit that similar triangles are involved, but they observe that no triangles in the diagram appear to be similar. At this time, they suspect that an auxiliary element needs to be constructed to reveal the elusive similar triangles.

The suggestion to construct, through C, a line parallel to $\overleftrightarrow{AB}$ that intersects lines $\overleftrightarrow{BM}$ and $\overleftrightarrow{AL}$ at points D and E, respectively (see **fig. 3**), prompts students to notice many pairs of similar triangles and then write all the possible proportions. For example, the similarity $\triangle ABL \sim \triangle ECL$ leads to $BL/CL = AB/EC = AL/EL$. Choosing the appropriate pairs of ratios that will verify the relationship $(AN/NB)(BL/LC)(CM/MA) = 1$ is challenging. Through discussion with team members and further guidance, some groups construct a proof. **Figure 3** displays such a proof.

To begin an indirect proof of the converse, we assume that the product equals 1 but that the cevians are not concurrent. This means that two cevians—say, $\overline{AL}$ and $\overline{BM}$—intersect at some point F but that the third one, $\overline{CN}$, does not go through F (see **fig. 4**). At this point, I usually provide the critical hint: Construct the line $\overleftrightarrow{CF}$ that by the crossbar theorem must

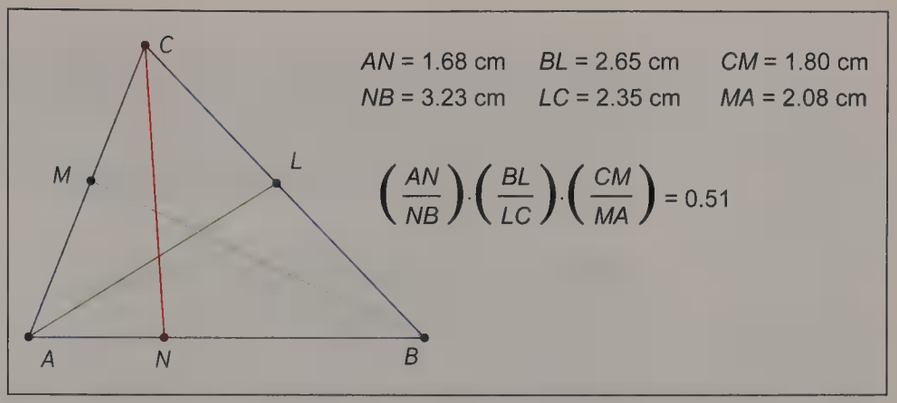


Fig. 1 Students explore this diagram with The Geometer's Sketchpad to discover Ceva's theorem.

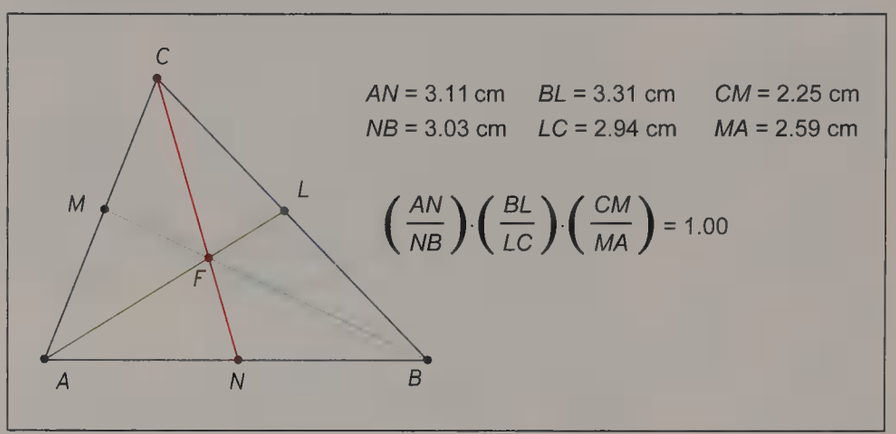
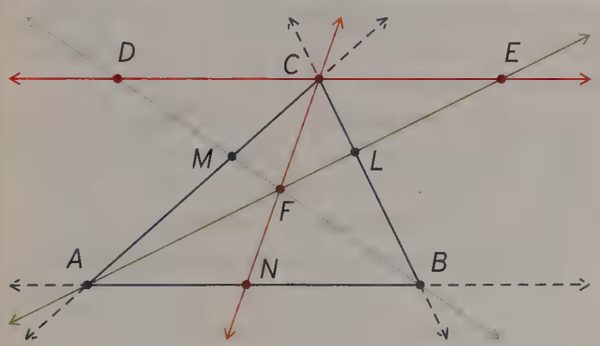


Fig. 2 The three cevians are concurrent when the product of division ratios is 1.



Assuming three concurrent cevians, we use angle-angle similarities to establish the following proportions:

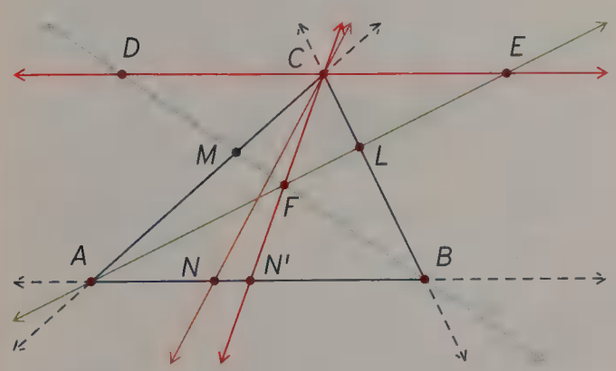
(*a*) $\triangle ABL \sim \triangle ECL \rightarrow BL/CL = AB/EC$
(*b*) $\triangle ABM \sim \triangle CDM \rightarrow CM/AM = CD/AB$
(*c*) $\triangle AFN \sim \triangle EFC \rightarrow AN/EC = FN/FC$
(*d*) $\triangle BFN \sim \triangle DFC \rightarrow BN/DC = FN/FC$

Then, (*c*) and (*d*) imply that $AN/BN = EC/DC$.

So

$$(AN/NB)(BL/LC)(CM/MA) = (EC/DC)(AB/EC)(CD/AB) = 1$$

Fig. 3 A proof of the first part of Ceva's theorem involves similarity of triangles.



. . . Since cevians $\overline{AL}$, $\overline{BM}$, and $\overline{CN'}$ of $\triangle ABC$ are concurrent, we can apply the first part of Ceva's theorem to prove that $(AN'/N'B) \cdot (BL/LC)(CM/MA) = 1$. On the other hand, we hypothesized that $(AN/NB)(BL/LC)(CM/MA) = 1$. From these two equalities, we have $AN'/N'B = AN/NB$. Therefore, with the collinearity of A, N, N', and B, we conclude that N is identical to N'. This contradicts our assumption that there is another line $\overline{CN'}$ that contains F. Hence, the three original cevians are concurrent.

Fig. 4 Students complete a proof of the second part of Ceva's theorem using an indirect method.

intersect segment $\overline{AB}$ at some point N' on the interior of $\overline{AB}$. Using this hint, students usually are able to complete the proof. The reasoning of one group of students is shown in **figure 4** and relies on the idea that the closer a point X on $\overline{AB}$ is to A, the smaller is the fraction AX/XB.

APPLICATIONS OF CEVA'S THEOREM

Once satisfied with the statement and proof of Ceva's theorem, students use the power of Ceva's theorem to prove several classical concurrency results. Students start their investigations in class but do most of the inquiry outside the classroom.

The *centroid*, the point of concurrency of the three medians of a triangle, is the perfect segue. The proof constructed by a group of students follows:

> Let $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ be the medians of $\triangle ABC$. According to Ceva's theorem, we need to prove that $(AF/FB)(BD/DC)(CE/EA) = 1$. But $(AF/FB) \cdot (BD/DC)(CE/EA) = (AF/AF)(BD/BD) \cdot (CE/CE) = 1$, so the three medians of a triangle are concurrent.

The class is usually amazed at the simplicity of the proof of the concurrency of the medians using Ceva's theorem.

Our next stop in our journey is at the *orthocenter*, the point of concurrency of a triangle's altitudes. With triangle similarity in mind, a group of students presented their proof (see **fig. 5**) and guided the class to choose the appropriate ratios of sides. One student in the group remarked that "it was like solving a puzzle."

Unfortunately, the "proof" in **figure 5** is fallacious, not to mention incomplete. It assumes that all the altitudes intersect at a point G. That was the proposition to be proved! Having students present and critique proofs is instructive. In this case, no student caught the essential circularity of the argument in **figure 5**. Their critiques focused on the right triangle and obtuse triangle cases, which were missing from the proof. Students using GSP noticed that, as they dragged vertex C, the similar triangles used in the proof seemed to disappear and, hence, the proof appeared to work only for acute triangles. With guidance, students can be led to valid proofs of the acute and obtuse triangle cases of the theorem, given in **figures 6** and **7**.

As we continue our trip, we stop at the *incenter* of a triangle, the point of intersection of the angle bisectors of the interior angles of a triangle, and the *circumcenter*, the point of intersection of the perpendicular bisectors of the sides of a triangle. In the case of the incenter, students invoke the interior angle bisector theorem to prove that the product of the division ratios is 1. This theorem states that an interior angle bisector divides the side opposite the bisected angle into segments proportional to the sides of the angle. Readers are invited to complete this proof.

In the case of the perpendicular bisectors of the sides of a triangle, students immediately remark that the requirements of Ceva's theorem are not satisfied because the perpendicular bisectors of the sides of a triangle are usually not cevians (see **fig. 8a**). I suggest to students that they construct the medial triangle of $\triangle ABC$ (see **fig. 8b**), and some always notice that the perpendicular bisectors of the sides of $\triangle ABC$ are the altitudes of its medial triangle. Since the altitudes of $\triangle DEF$ are concurrent, so are the perpendicular bisectors of $\triangle ABC$.

Another surprising point of concurrency that

. . . Using angle-angle similarities (because vertical angles are congruent and right angles are associated with each altitude), we can establish the following result:

$$\triangle ADG \sim \triangle CEG \rightarrow AD/CE = DG/EG$$
$$\triangle BDG \sim \triangle CFG \rightarrow CF/BD = FG/DG$$
$$\triangle BEG \sim \triangle AFG \rightarrow BE/AF = EG/FG$$

Hence,

$$\begin{aligned}(AD/DB)(BE/EC)(CF/FA) &= (AD/EC)(BE/FA)(CF/DB)\\ &= (DG/EG)(EG/FG)(FG/DG)\\ &= 1.\end{aligned}$$

Therefore, the three altitudes are concurrent by Ceva's theorem.

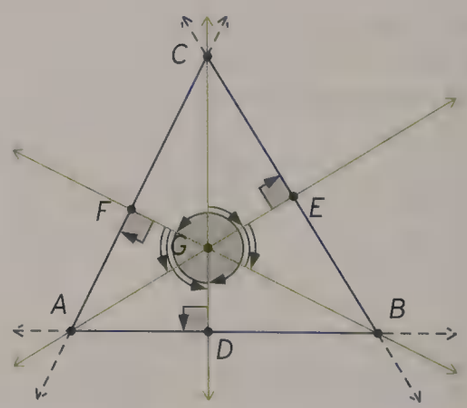
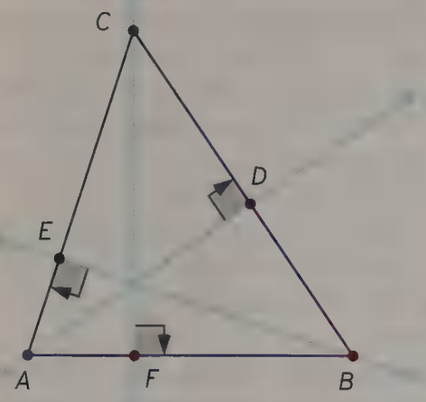


Fig. 5 A circular argument has the altitudes intersect at G.



Let $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ be the altitudes of an acute triangle $\triangle ABC$.

Using angle-angle similarity (for right triangles that share a common angle), we can establish the following results:

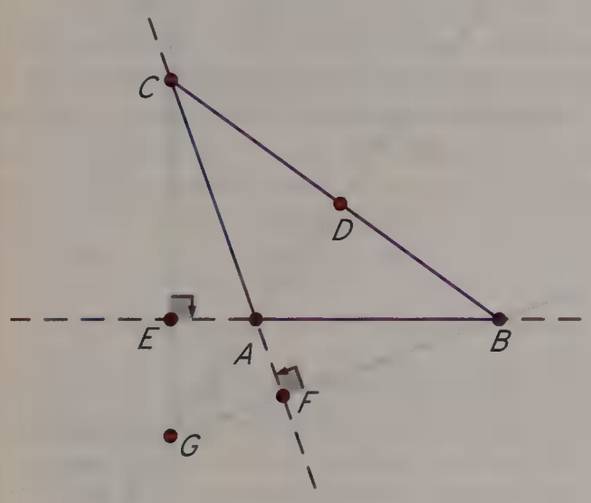
$\triangle BCE \sim \triangle ACD \rightarrow CE/CD = BC/AC.$
$\triangle ACF \sim \triangle ABE \rightarrow AF/AE = AC/AB$
$\triangle ABD \sim \triangle CBF \rightarrow BD/BF = AB/BC.$

Thus,

$$\begin{aligned}(AF/FB)(BD/DC)(CE/EA) &= (AF/EA)(BD/FB)(CE/DC) \\ &= (AC/AB)(AB/BC)(BC/AC) \\ &= 1.\end{aligned}$$

In other words, the three altitudes of an acute triangle are concurrent by Ceva's theorem.

Fig. 6 The altitudes of an acute triangle are concurrent at the orthocenter.



Let $\triangle ABC$ be a triangle with an obtuse angle—say, $\angle CAB$.

Assume that lines containing altitudes $\overline{BF}$ and $\overline{CE}$ intersect at G exterior to $\triangle ABC$. Triangle BCG is acute. (Why?)

Since $\overline{BE}$ and $\overline{CF}$ intersect at A and are altitudes of $\triangle BCG$, then the line $\overleftrightarrow{GA}$ contains the third altitude of $\triangle BCG$ because altitudes of an acute triangle are concurrent. $\overleftrightarrow{GA}$ is therefore perpendicular to $\overleftrightarrow{BC}$ at D. Hence, all three (extended) altitudes of $\triangle ABC$ contain G.

Fig. 7 The extended altitudes of an obtuse triangle are concurrent.

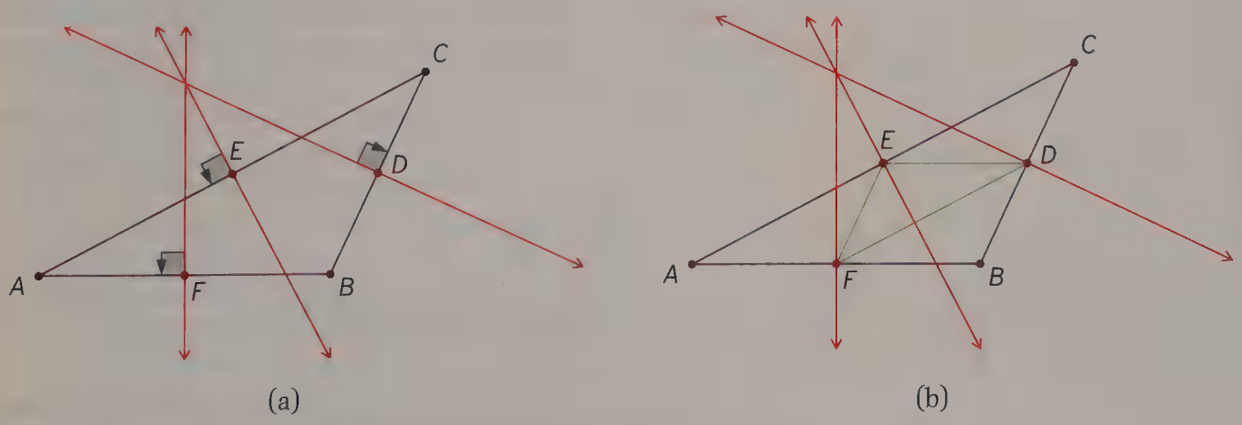


Fig. 8 The perpendicular bisectors of the sides of a triangle are concurrent.

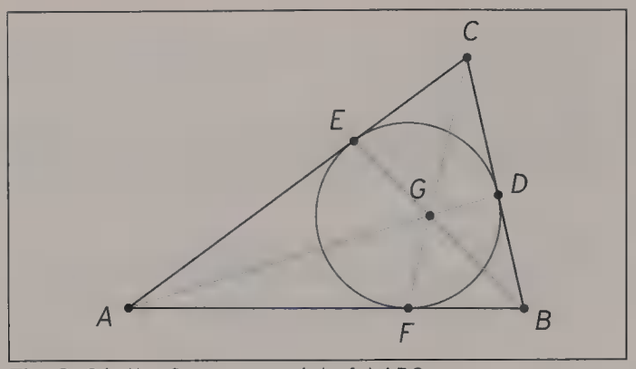


Fig. 9 *G* is the Gergonne point of $\triangle ABC$.

I have students explore—the Gergonne point—is obtained by using points of tangency of the incircle as the division points of the three sides (see **fig. 9**). The proof is based on the theorem that if tangents at points E and F on a circle intersect at point A, then $AE = AF$. Thus, as shown in **figure 9**, $BD = BF$ and $CD = CE$, implying that

$$(AF/FB)(BD/DC)(CE/EA) = (AF/FB)(BF/DC)(CD/AF) = 1.$$

The class continues visiting other centers of a triangle to view their existence through the lens of Ceva's theorem (e.g., the Nagel point, the Lemoine point, and the Fermat point all have many interesting properties that can lead to challenging proofs; see Kay 2001). Their adventures are typically summarized by observations such as this one made by a student after proving the concurrency of medians of a triangle: "Both the synthetic and the analytic proof of the concurrency of the medians are more intricate and cumbersome than this trivial application of Ceva's theorem."

EXTENSIONS OF CEVA'S THEOREM

Specializing, generalizing, and extending theorems are common activities in my geometry course. One avenue is to extend Ceva's theorem to polygons with more sides. Students use GSP to experiment with polygons and are delighted to discover on their own that Ceva's theorem can be extended to polygons with an odd number of sides. For example, if the cevians through the vertices A, B, C, D, and E of a convex pentagon intersect the opposite sides at A', B', C', D', and E', respectively, and the cevians are concurrent, then

$$(AD'/D'B)(BE'/E'C)(CA'/A'D)(DB'/B'E)(EC'/C'A) = 1$$

(see **fig. 10**). Ceva's theorem is certainly a box full of surprises!

Another activity has students investigate and discuss several alternative proofs. With Ceva's theorem, one proof that I like avoids the use of auxiliary elements and is based on a strategy involving areas instead of similar triangles. To help students develop this strategy, I ask them to represent one of the ratios in the equation $(AN/NB)(BL/LC)(CM/MA) = 1$ in terms of the ratio of triangle areas. Most students find AN/NB as $area(\triangle ACN)/area(\triangle NCB)$ and $area(\triangle AFN)/area(\triangle NFB)$ (see **fig. 2**) because the ratio of areas of two triangles with a common altitude is the ratio of the lengths of their bases, a theorem that the class previously proved. Finding the third pair of triangles is more elusive, so I suggest a basic property of proportions:

$$\frac{p}{q} = \frac{r}{s} = \frac{p-r}{q-s}.$$

At this point, some students recognize that

$$\frac{AN}{NB} = \frac{area(\triangle ACN) - area(\triangle AFN)}{area(\triangle NCB) - area(\triangle NFB)} = \frac{area(\triangle ACF)}{area(\triangle BCF)}.$$

Reasoning by analogy, the class concludes that $BL/LC = area(\triangle BAF)/area(\triangle CAF)$ and $CM/MA = area(\triangle CBF)/area(\triangle ABF)$. We can now simply make appropriate substitutions to obtain

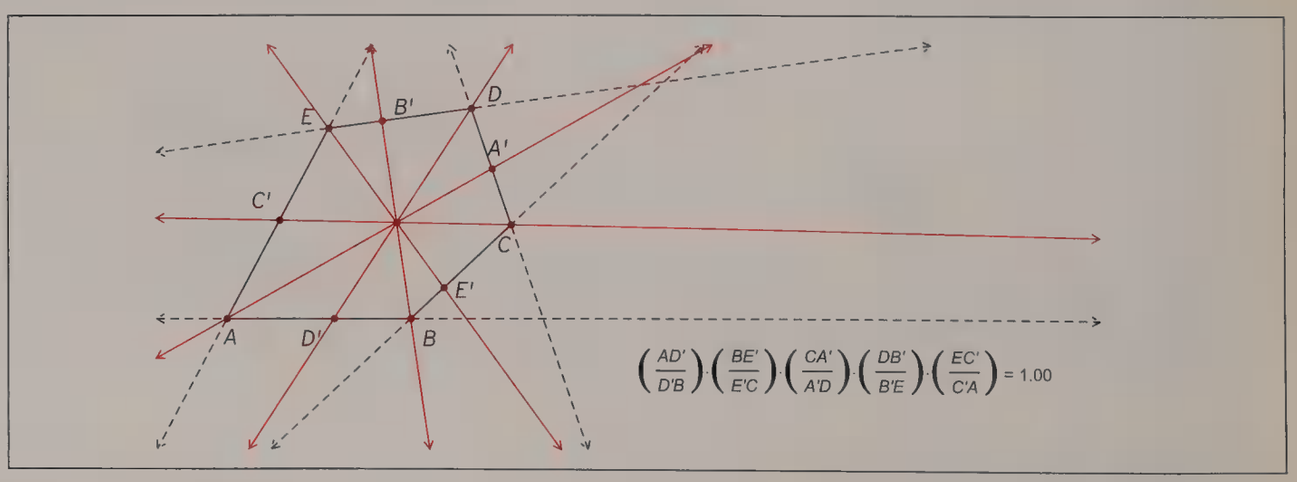
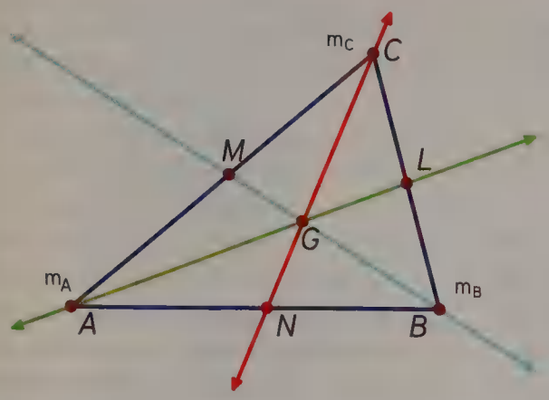


Fig. 10 Ceva's theorem can be extended to polygons with an odd number of sides.



Let $\overline{AL}$, $\overline{BM}$, and $\overline{CN}$ be cevians of $\triangle ABC$ concurrent at G. Place masses m_A, m_B, and m_C at points A, B, and C, respectively, such that N is the centroid of m_A and m_B and L is the centroid of m_B and m_C. Thus, $AN/NB = m_B/m_A$, and $BL/LC = m_C/m_B$. We can replace masses m_A and m_B by one mass, $m_A + m_B$ at N, without affecting the equilibrium of the system. Therefore, the centroid of the three masses is located on $\overline{CN}$. Similarly, the centroid of the three masses is located on $\overline{AL}$. Hence, the centroid of the three masses is G, the point of intersection of $\overline{CN}$ and $\overline{AL}$. Now, the segment connecting the centroid of m_A and m_C to B must also contain the centroid of the three masses—that is, G. Hence, the centroid of m_A and m_C must be the intersection of line BG with $\overline{AC}$, which is M. Thus, $CM/MA = m_A/m_C$, yielding $(AN/NB)(BL/LC)(CM/MA) = (m_B/m_A)(m_C/m_B)(m_A/m_C) = 1$.

Fig. 11 Ceva's theorem can be proved using an argument based on mechanics.

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} =$$

$$\frac{area(\triangle ACF)}{area(\triangle BCF)} \cdot \frac{area(\triangle BAF)}{area(\triangle CAF)} \cdot \frac{area(\triangle CBF)}{area(\triangle ABF)} = 1.$$

Another proof that my students find interesting because of its historical nature and ties to physics is akin to the proof of the theorem originally given by Ceva. The justification of the first part of Ceva's theorem that is displayed in **figure 11** involves the following laws of mechanics:

- The centroid of two (nonzero) masses m_A and m_B placed at points A and B, respectively, is the point N between A and B such that $m_A \cdot AN = m_B NB$—that is, $AN/BN = m_B/m_A$.
- The equilibrium of a system does not change if we replace two masses m_A and m_B by a mass $m_A + m_B$ placed at their centroid.

A THEOREM TO APPRECIATE

Some of the proofs of Ceva's theorem involve basic geometric concepts (e.g., similarity of triangles and area), so it is surprising that some college geometry textbooks for future mathematics teachers do not discuss this important theorem. Because the principles behind Ceva's theorem are studied in any high school geometry class, this theorem and the applications discussed here are within the reach of a broad audience. The theorem can also serve as a source of enrichment topics for advanced high school students. I myself was delighted when I learned this surprising theorem and one of its proofs in my college geometry course. Using Ceva's theorem in our geometry classes allows us to appreciate the elegance, power, and ubiquity of mathematical ideas.

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JOSÉ N. CONTRERAS, jncontrerasf@bsu.edu, teaches mathematics and mathematics education courses at the undergraduate and graduate levels at Ball State University in Muncie, Indiana. He is interested in integrating problem posing and solving, technology, realistic contexts, modeling, history, proof, and aesthetic aspects of mathematics in teaching and teacher education.

FOR YOUR information

FROM OTHER PUBLISHERS

Beautiful Geometry, Eli Maor and Eugen Jost, 2014. 208 pp., $27.95 cloth. ISBN 978-0-691-15099-4. Princeton University Press; www.press.princeton.edu.

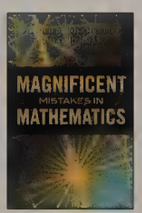
This delightful book takes fifty geometry topics, giving a brief history of each, and illustrates geometric patterns and theorems with more than sixty color plates (whole-page illustrations printed separately from the text). The aim is to "reach a broad audience of high school and college students, mathematics and science teachers, university instructors, and laypersons who are not afraid of an occasional formula or equation." In general, each topic consists of one to three pages of information and background, accompanied by illustrations and color plates. Topics include elementary geometry, such as the Pythagorean theorem, quadrilaterals, and the square root of 2, as well as topics such as the Euler line, Lissajous figures, and Sierpinski's triangle.

One of the topics, perfect numbers and triangular numbers, includes illustrations and descriptions of the five Platonic solids as an example of the sacred status that the Greeks accorded the number five. This introduction leads to the number six, which was "even more revered" as the "first perfect number, being the sum of its proper divisors." This mention of six, also a triangular number, leads into a discussion of equations and relations, relations that the Pythagoreans viewed as "purely geometric," whereas we tend to prove them algebraically. The accompanying color plate provides several "dots arranged in various geometric patterns."

I recommend this book for those who enjoy the integration of mathematics and art as an expression of pattern and order.

—Marsha Guntharp
Palm Beach Atlantic University
West Palm Beach, FL

Magnificent Mistakes in Mathematics, Alfred S. Posamentier and Ingmar Lehmann, 2013. 296 pp., $24.00 cloth. ISBN 978-1-61614-747-1. Prometheus Books; www.prometheusbooks.com.



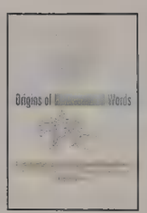
Designed for those with knowledge of at least high school mathematics, this intriguing work discusses how anyone can make mathematical mistakes, including mistakes that remained elusive for ages before being discovered and corrected.

The book focuses on the centrality of using critical thinking when solving problems—for example, realizing that rounding down when less than "halfway" does not always work and that dividing by zero never works. This account concentrates on twenty-five mistakes by famous mathematicians, such as Pythagoras's incorrect belief that only natural numbers can measure a regular pentagram's dimensions, mistakes in proving Fermat's last theorem, and developmental errors in the four-color map problem. Also discussed are several mistakes from arithmetic (e.g., using an incorrect procedure to correctly multiply fractions); algebra (e.g., dividing by zero to show that 1 = 2; my students enjoyed this example and gained a better understanding as to why dividing by zero does not work); geometry (e.g., assuming that the angular sum is identical for all triangles when trying to prove that the sum equals 180 degrees); and probability and statistics (e.g., the need to think first before starting a solution).

By understanding some of the mistakes made in its development, readers will gain a greater appreciation of mathematics as well as become better critical thinkers. This is a wonderful exploration that is appropriate for its intended audience.

—Whiting Wicker
Valencia Community College
Orlando, FL

Origins of Mathematical Words: A Comprehensive Dictionary of Latin, Greek, and Arabic Roots, Anthony Lo Bello, 2013. 350 pp., $49.95 paper. ISBN 978-1-4214-1098-2. Johns Hopkins University Press; www.press.jhu.edu.



The language of mathematics requires a level of precision to which mathematicians and mathematics educators must adhere to communicate mathematics effectively. Although several mathematical dictionaries are available to help anyone learn the meaning of mathematical words, Lo Bello's book presents an etymological approach. His goal is to provide the historical origins, form, and usage of these words, specifically those whose roots are Latin, Greek, or Arabic. The author purposefully asserts that correct form and usage of mathematical words are crucial to avoid the degradation of language itself.

The entries are organized alphabetically, and each includes the historical Latin, Greek, or Arabic roots. Entries may also include definitions, uses in literary sources (such as well-known mathematical texts, personal correspondence, poetry, written commentary, and

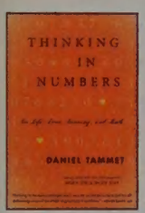
Prices of software, books, and materials are subject to change. Consult the suppliers for the current prices. The comments reflect the reviewers' opinions and do not imply endorsement by the National Council of Teachers of Mathematics.

dictionaries), historical and modern uses, and examples of incorrect uses. Several entries also include symbolic mathematical examples.

This book is a reliable resource for mathematics educators, mathematics historians, graduate students in the mathematical sciences, and anyone interested in the origins of many common mathematical words. I recommend it to anyone who wishes to learn, use, and understand correct mathematical terminology based on historical origins and meanings.

—Kelli M. Slaten
University of North Carolina Wilmington

Thinking in Numbers: On Life, Love, Meaning, and Math, Daniel Tammet, 2013. 288 pp., $26.00 cloth. ISBN 978-0-316-18737-4. Little, Brown, and Company; www.littlebrown.com.



Thinking in Numbers is a delightful read for anyone, whether interested in mathematics or not, but it should be read purely for enjoyment. It would not work well as a supplement for a mathematics course or as assigned reading for students.

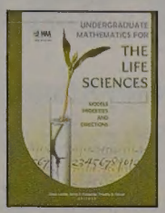
Tammet is a well-established author with two previously published books; both gained critical acclaim. He also is well known as an autistic savant and sought after as a research subject because, unlike most savants, he is quite articulate about what he can do. His first two books described how he manages life as a person on the high-functioning end of the autism spectrum. This book describes his relationship with mathematics.

Tammet is an eloquent writer, whether he is describing finitely large numbers or explaining how many Icelandic words represent the numbers one through four (more than fifty, all of which Tammet knows). The essays are at times analytic, autobiographical, historical, and aesthetic. Because each chapter can stand alone and most are about ten pages long, the book is ideal for picking up at odd moments when one has a few minutes to read.

I thoroughly enjoyed this book and have added Tammet's other two books to my list of future reads.

—Albert Goetz
George Mason University (adjunct)
Annandale, VA

Undergraduate Mathematics for the Life Sciences: Models, Processes, and Directions, Glenn Ledder, Jenna P. Carpenter, Timothy D. Comar, eds., 2013. 207 pp., $25.00 eBook/PDF, ISBN 978-1-6144-316-2; $43.00 POD, ISBN 978-0-88385-191-3. Mathematical Association of America; www.maa.org.



This collection of essays summarizes the design and implementation of legitimate courses from various universities around the country that integrate mathematics and biology. Each essay illustrates the structure of the course, the reason for such design, and how students explored mathematics through science. The book targets the professor or department chair at the university level who wishes to implement a course with strong connections between mathematics and sciences and is a great resource for those interested in or affiliated with STEM.

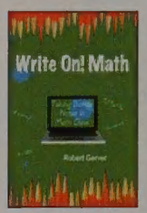
This book superbly demonstrates how the typical mathematics professor can use real-life science contexts to teach mathematics concepts and engage students who are not majoring in mathematics. In addition, the authors of each essay narrate their findings after implementing the course and indicate changes that could be made moving forward. This feedback is particularly helpful for anyone interested in attempting to teach courses similar to these.

Most of the course designs are progressive and have deep connections to mathematics, but the course entitled "A Terminal Post-Calculus-I Mathematics Course for Biology Students" (chapter 7) contradicts this premise. This author talks a lot about the idea of mathematical modeling; however, the approach is somewhat limited because it seems to view mathematical modeling as simply creating and revising functions.

I highly recommend this book to any department looking to design a course in which mathematics is taught in a life science context.

–Edward DePeau
Central Connecticut State University
New Britain, CT

Write On! Math, Robert Gerver, 2013. 87 pp., $29.99 paper. ISBN 978-1-62396-406-1. Information Age Publishing; www.infoagepub.com.



These educational and organizational "tips," emphasizing organization and detail, teach students structured ways of taking notes, expanding on their notes later, and writing a structured mathematical proof or research paper. The author is an experienced high school mathematics teacher who has had success using these techniques to improve students' success in a traditional mathematics classroom setting. The information could be useful for students who can achieve a deep mathematical understanding solely from lectures and time spent alone expanding on their notes.

However, this book fails to challenge traditional views of teaching mathematics and would not help many struggling students make sense of challenging new content. The author assumes that mathematics is taught through lectures as students quietly listen to their teacher talk or write at chalkboard, whiteboard, or SMART Board™. The author's philosophy does not seem to align with social constructivist views of learning or research-based strategies for teaching mathematics.

This book may be useful to unfortunate students placed in a traditional mathematics classroom (as many students are). It would not be relevant, however, to students who are experiencing high-quality mathematics instruction and are actively engaged in making and refining conjectures with peers. It does not appear to give students the opportunity to experience a redefined view of what mathematics is.

—Lindsay Keazer
Michigan State University
East Lansing, MI

MY FAVORITE lesson

Daniel A. Showalter

Dot Tag and Representational Fluency

Teaching precalculus can be a special challenge because students must integrate algebra and geometry into a connected view of mathematics. The Common Core State Standards for Mathematics call for connections among mathematical fields, but developing tasks that integrate algebraic and graphical representations can be time-consuming. One of my favorite activities, Dot Tag, requires only minutes to set up and encourages students to build on what they have learned about graphing algebraic functions in the Cartesian plane.

Dot Tag is a simplified version of the computer game Green Globs (Dugdale and Kibbey 2013). A collection of points (dots) is graphed, and student teams take turns generating functions whose graphs intersect as many dots as possible. **Figure 1** is an example; the line with equation $y = 0.54x + 1.9$ intersects 3 of the 8 dots. I have used variations: restricting students to certain types of functions, such as linear or quadratic, or perhaps to nonpolynomial functions in an AP Calculus class; allowing plots of nonfunction relations such as circles; weighting specific dots to be of higher or lower value; increasing the dot size to allow for greater tolerance; and plotting "negative dots" that, when intersected, require that points be subtracted from a team's overall score.

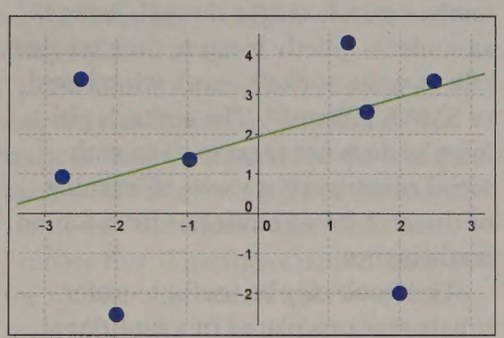


Fig. 1 A line intersects three dots in GeoGebra 5.0.

I enjoy using Dot Tag in my precalculus classes for several reasons:

- *Dot Tag teaches representational fluency.* Representational fluency is the ability to translate across representations, gain understanding about the underlying entities that are being represented, and generalize across representations (Zbiek et al. 2007). To succeed in Dot Tag, students must translate between graphical and algebraic representations.
- *Dot Tag is a high-level cognitive task.* Engaging students in high-level cognitive tasks is a crucial component for developing mathematical reasoning (Stein et al. 2009). Not only does Dot Tag build representational fluency, it also presents students with the open-ended problem of determining *which* dots to try to pass through.
- *Dot Tag is effective with students at many levels.* I have used Dot Tag with students and teachers of wide-ranging mathematical ability. Each participant found it challenging at his or her own level of mathematical understanding, whether generating a function passing through two dots or creating a matrix to solve a system of fifth-degree polynomials. The lesson differentiates instruction, a difficult job. (See Yang and Gordon [2014] to find a polynomial passing through n points.)
- *Dot Tag requires little preparation.* Open a graphing application, such as GeoGebra, click on the **Point** tool, and plot several points. Because Dot Tag can be adapted for use with most precalculus topics, I keep it as an emergency lesson to use when I am pressed for time.

I love Dot Tag because students love it. While playing a game, my students are developing representational fluency. I feel no guilt yielding the floor to the dots!

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The Back Page provides a forum for readers to share a favorite lesson, not exceeding 600 words. Lessons should be submitted to mt.msubmit.net.

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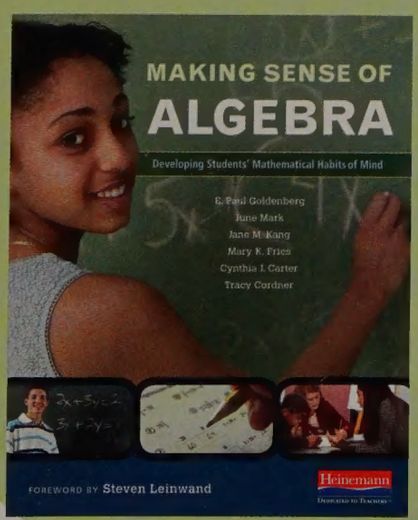

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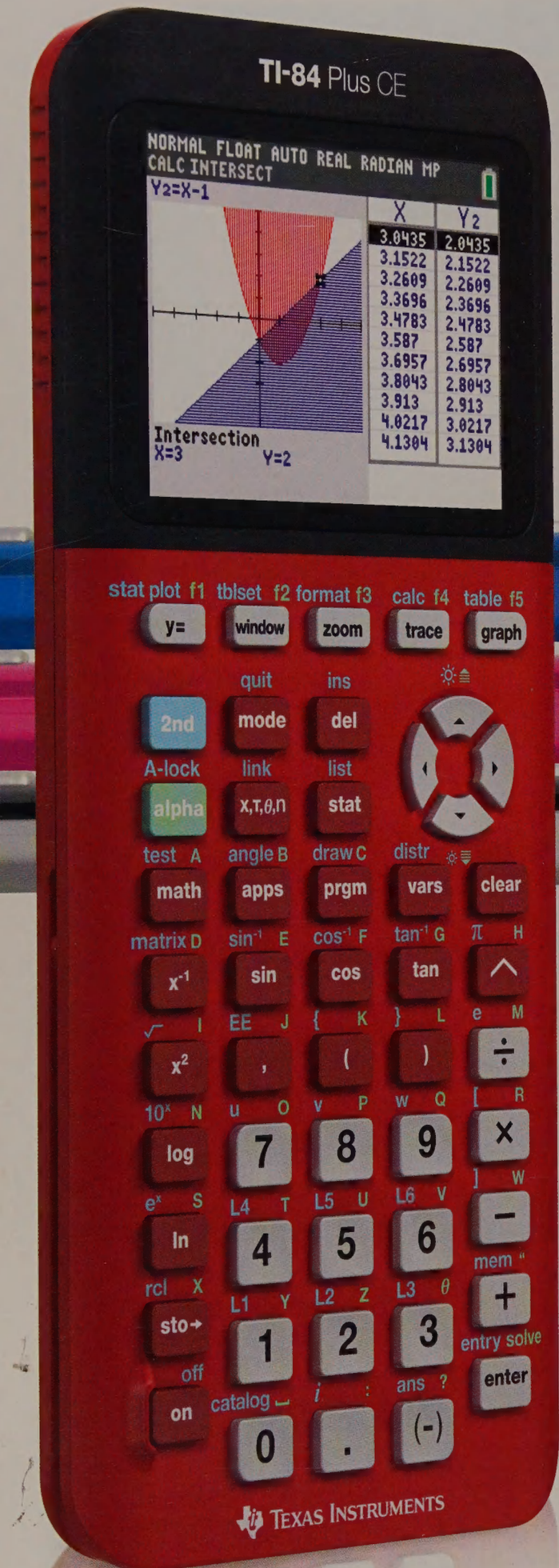
TI-84 Plus CE
NORMAL FLOAT AUTO REAL RADIAN MP
CALC INTERSECT
Y2=X-1
X Y2
3.0435 2.0435
3.1522 2.1522
3.2609 2.2609
3.3696 2.3696
3.4783 2.4783
3.587 2.587
3.6957 2.6957
3.8043 2.8043
3.913 2.913
4.0217 3.0217
4.1304 3.1304
Intersection
X=3 Y=2
stat plot f1 tblset f2 format f3 calc f4 table f5
y= window zoom trace graph
quit ins
2nd mode del
A-lock link list
alpha x,T,θ,n stat
test A angle B draw C distr
math apps prgm vars clear
matrix D sin⁻¹ E cos⁻¹ F tan⁻¹ G π H
x⁻¹ sin cos tan ^
√ I EE J { K } L e M
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10^x N u O v P w Q [R
log 7 8 9 ×
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ln 4 5 6 −
rcl X L1 Y L2 Z L3 θ mem "
sto→ 1 2 3 +
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